Algebraic relations and Boij-Söderberg theory

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Abstract

One of the common invariants of a graded module over a graded commutative ring is the Betti number. For any graded minimal free resolution $F_\bullet$ of a graded R-module, we have corresponding Betti numbers that record information about the grading of $F_\bullet$. Using a specific index, we can construct a Betti diagram with Betti numbers as entries. Inspired by a set of conjectures of M. Boij and J. Söderberg, an algorithm was given by D. Eisenbud and F. Schreyer allowing the decomposition of Betti diagrams into pure diagrams. In this thesis, we explore the basic concepts of Boij-Söderberg theory, including the construction of minimal free resolutions of graded R-modules, Betti diagrams, and Betti decomposition. We investigate the relationship between the Betti decompositions of graded R-modules that form a short exact sequence and find that there is a class of short exact sequences of modules such that the Betti decomposition of the middle module is equivalent to the sum of the Betti decompositions of the outer two modules. We also examine the decompositions of Betti diagrams over a special kind of ring called a complete intersection, which furthers the results of C. Gibbons, J. Jeffries, S. Mayes, C. Raicu, B. Stone, B. White (2012) [6] to codimension 4.

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Dedication

To Amalie Emmy Noether, for your invaluable contributions to abstract algebra, without which this project would not exist. You are an inspiration to all young women in mathematics.

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I would like to thank the following people: Branden Stone, for your time, encouragement, and unlimited patience throughout the past year; Eileen Brickner, for your unwavering emotional and moral support; Molly, Mom, Dad, Grandma, and Grandpa, for always having faith in me. Thank you for helping me to create something that I can really be proud of.

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Introduction.

1.1 History and background

In 2008, M. Boij and J. Söderberg published an article [2] describing two conjectures relating to the Betti diagram representations of graded free resolutions of Noetherian modules. One of these conjectures, restated in this paper with Theorem 1.4.4, views Betti diagrams as sitting inside some vector space, and as a result they can be written as linear combinations of basis elements. In 2009, D. Eisenbud and F. Schreyer proved Theorem 1.4.4 using an algorithm that "decomposes" Betti diagrams into linear combinations of basis elements. This algorithm is one of the main tools we employ in order to study the Betti diagrams of Noetherian modules.

The study of Noetherian modules would not be possible without the groundbreaking research of the German mathematician Emmy Noether in the 1920s. One of Noether's most important contributions to abstract algebra was her clever use of ascending (or descending) chain conditions. Any object in abstract algebra satisfying these conditions is now referred to as "Noetherian" in her honor. A module M is Noetherian if it satisfies the ascending chain conditions on its submodules, or, equivalently, if every submodule of M

is finitely generated. The following remark comes from the theory of Noetherian modules. It is well known that this is the case.

Remark 1.1.1. [1] If M is a finitely generated module over $R = k[x_1, \ldots, x_n]$, then every submodule $N \subseteq M$ is finitely generated. $\Diamond$

We will use this property of finitely generated modules when constructing the graded minimal free resolution of a graded R module in Section 1.3.

1.2 Motivation and basic definitions.

The two main questions that we will explore in this paper are as follows.

Question 1.2.1. *Let R be a ring. Consider a short exact sequence of R-modules:*

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0.$$

Given the Betti decompositions of A and C, what can we conclude about the Betti decomposition of B?

Question 1.2.2. *Let $S = k[x_1, \ldots, x_d]$ be a polynomial ring over a field k and let $I = (f_1, \ldots, f_d)$ be an ideal of S generated by a homogeneous regular sequence with deg$(f_i) = e_i$. What is the Betti decomposition of S/I in terms of the degrees e_i?*

To fully understand these questions, we need some background material. We will discuss rings, ideals, modules, short exact sequences, and standard grading in Section 1.2. In Section 1.3, we will introduce generating sets and free modules, which will lead to the construction of minimal graded free resolutions. In Section 1.4, we will explore Betti diagrams, each of which is unique to a given minimal graded free resolution. We will revisit Question 1.2.1 in Chapter 2 and Question 1.2.2 in Chapter 3 once we have all the necessary tools.

[1] See any introduction to commutative algebra text, such as [1], [4], [11], [12]

Definition 1.2.3. A *ring* is a set R with binary operators $(+, \cdot)$ such that R is an abelian group under addition and has the following properties:

1. $x \cdot y \in R$,

2. $(x \cdot y) \cdot z = x \cdot (y \cdot z)$,

3. $x \cdot (y + z) = x \cdot y + x \cdot z$,

for all $x, y, z \in R$. A ring R is *commutative* if $x \cdot y = y \cdot x$ for all $x, y \in R$. A ring R has a multiplicative identity $1_R \in R$ if $1_R \cdot x = x \cdot 1_R = x$ for all $x \in R$. △

In this paper, we will assume all rings are commutative and include a multiplicative identity.

Example 1.2.4. The sets $\mathbb{Z}, \mathbb{R}, \mathbb{Q}$, and $\mathbb{C}$ are all commutative rings with multiplicative identity. ♢

Example 1.2.5. Let k be a field. Define $R = k[x]$ as the set of polynomials of the form $\sum_{i \geq 0} c_i x^i$ such that $c_i \in k$. We have that $3 - x, 7x^3, 2x + x^2 \in R$. Since k is a field, then the sum of any two polynomials in R will still be in one variable with coefficients in k. Observe that $(2x + x^2) + (3 - x) = 3 + x + x^2 \in R$ and $(3 - x) + (7x^3) = 3 - x + 7x^3 \in R$. Using this technique, it is easy to show that R is closed. Thus R is closed under addition. Note that each element $r(x) \in R$ has an additive inverse, given by $-r(x)$. Since addition of polynomials is commutative, it follows that R is an abelian group under addition. We will go through the three properties of a ring from Definition 1.2.3 to show that R is a ring. Let $r(x), s(x), t(x) \in R$. Then we can write

$$r(x) = \sum_{i=0}^{n} r_i x^i,$$
$$s(x) = \sum_{j=0}^{m} s_j x^j,$$
$$t(x) = \sum_{k=0}^{l} t_k x^k,$$

for $r_i, s_j, t_k \in k$. Then we have that

$$\begin{aligned} r(x) \cdot s(x) &= \left(\sum_{i=0}^{n} r_i x^i\right)\left(\sum_{j=0}^{m} s_j x^j\right) \\ &= \sum_{i=0}^{n}\sum_{j=0}^{m} r_i s_j x^{i+j}. \end{aligned}$$

Since $r_i, s_j \in k$ and k is a field, then $r_i s_j \in k$ for all $0 \le i \le n, 0 \le j \le m$. By our definition of R, it follows that $r(x) \cdot s(x) \in R$. Consider $3 - x, 7x^3 \in k[x]$. Multiplying these two polynomials together, we get $(3 - x)(7x^3) = 21x^3 - 7x^4$. Since $21, 7 \in k$, it follows that $21x^3 - 7x^4 \in k[x]$.

We also have that

$$\begin{aligned} (r(x) \cdot s(x)) \cdot t(x) &= \left(\left(\sum_{i=0}^{n} r_i x^i\right) \cdot \left(\sum_{j=0}^{m} s_j x^j\right)\right) \cdot \left(\sum_{k=0}^{l} t_k x^k\right) \\ &= \left(\sum_{i=0}^{n}\sum_{j=0}^{m} r_i s_j x^{i+j}\right) \cdot \left(\sum_{k=0}^{l} t_k x^k\right) \\ &= \sum_{i=0}^{n}\sum_{j=0}^{m}\sum_{k=0}^{l} r_i s_j t_k x^{i+j+k} \\ &= \left(\sum_{i=0}^{n} r_i x^i\right) \cdot \left(\sum_{j=0}^{m}\sum_{k=0}^{l} s_j t_k x^{j+k}\right) \\ &= r(x) \cdot (s(x) \cdot t(x)). \end{aligned}$$

Consider $f(x) = 3 - x, g(x) = 7x^3, h(x) = 2x + x^2 \in k[x]$. We find that

$$\begin{aligned}(f(x) \cdot g(x)) \cdot h(x) &= ((3-x)7x^3)(2x + x^2) \\ &= (21x^3 - 7x^4)(2x + x^2) \\ &= 42x^4 + 7x^5 - 7x^6,\end{aligned}$$

and that

$$\begin{aligned}f(x) \cdot (g(x) \cdot h(x)) &= (3-x)(7x^3(2x + x^2)) \\ &= (3-x)(14x^4 + 7x^5) \\ &= 42x^4 + 7x^5 - 7x^6.\end{aligned}$$

So $(f(x) \cdot g(x)) \cdot h(x) = f(x) \cdot (g(x) \cdot h(x))$.

We define $s_j = 0$ for $j > m$. For the final property in Definition 1.2.3, we have

$$\begin{aligned}r(x) \cdot (s(x) + t(x)) &= \left(\sum_{i=0}^{n} r_i x^i\right) \cdot \left(\left(\sum_{j=0}^{m} s_j x^j\right) + \left(\sum_{k=0}^{l} t_k x^k\right)\right) \\ &= \left(\sum_{i=0}^{n} r_i x^i\right) \cdot \left(\sum_{j=0}^{\max(m,l)} (s_j + t_j) x^j\right) \\ &= \sum_{i=0}^{n} \sum_{j=0}^{\max(m,l)} r_i (s_j + t_j) x^{i+j} \\ &= \sum_{i=0}^{n} \sum_{j=0}^{\max(m,l)} (r_i s_j + r_i t_j) x^{i+j} \\ &= \left(\sum_{i=0}^{n} \sum_{j=0}^{m} r_i s_j x^{i+j}\right) + \left(\sum_{i=0}^{n} \sum_{j=0}^{l} r_i t_j x^{i+j}\right) \\ &= r(x) \cdot s(x) + r(x) \cdot t(x).\end{aligned}$$

Again, consider the three polynomials

$$\begin{aligned}f(x) &= 3 - x, \\ g(x) &= 7x^3, \\ h(x) &= 2x + x^2,\end{aligned}$$

in $k[x]$. Notice that

$$f(x) \cdot (g(x) + h(x)) = (3 - x)(7x^3 + 2x + x^2) = 20x^3 + 6x + x^2 - 7x^4$$

and

$$f(x) \cdot g(x) + f(x) \cdot h(x) = (3 - x)7x^3 + (3 - x)(2x + x^2) = 20x^3 - 7x^4 + 6x + x^2.$$

It follows that $f(x) \cdot (g(x) + h(x)) = f(x) \cdot g(x) + f(x) \cdot h(x)$.

Observe that

$$\begin{aligned} r(x) \cdot s(x) &= \sum_{i=0}^{n} \sum_{j=0}^{m} r_j s_j x^{i+j} \\ &= \sum_{j=0}^{m} \sum_{i=0}^{n} s_j r_i x^{j+i} \\ &= s(x) \cdot r(x), \end{aligned}$$

so R is commutative.

Notice that $1 \cdot r(x) = r(x) \cdot 1 = r(x)$ for all $r(x) \in R$. Thus we have shown that $R = k[x]$ is a commutative ring with a multiplicative identity. $\Diamond$

Proposition 1.2.6. *Let R be a commutative ring. Then*

$$R[x] := \left\{ \sum_{i=0}^{n} r_i x^i \mid r_i \in R \right\}$$

is a ring.

Proof. Let R be a ring. Consider the set defined by

$$R[x] := \left\{ \sum_{i=0}^{n} r_i x^i \mid r_i \in R \right\}.$$

Let $r, s, t \in R[x]$. Then

$$r = \sum_{i=0}^{n} r_i x^i,$$
$$s = \sum_{i=0}^{m} s_i x^i,$$
$$t = \sum_{i=0}^{l} t_i x^i,$$

for $r_i, s_i, t_i \in R$.. Observe that we can employ the same argument as in Example 1.2.5 to show that

$$r \cdot s \in R,$$
$$(r \cdot s) \cdot t = r \cdot (s \cdot t),$$
$$r \cdot (s + t) = r \cdot s + r \cdot t,$$
$$r \cdot s = s \cdot r,$$
$$\text{and } 1 \cdot r = r. \quad \square$$

We define

$$R[x_1, \ldots, x_n] := \left\{ \sum_{v_i \geq 0} r_{\underline{v}} \underline{x}^{\underline{v}} \mid \underline{v} \in \mathbb{Z}^n \right\}$$

as the set of polynomials over R in n variables, where $\underline{x} = [x_1, \ldots, x_n], \underline{v} = [v_1, \ldots, v_n]$ are vectors and $\underline{x}^{\underline{v}} = x_1^{v_1} x_2^{v_2} \cdots x_n^{v_n}$.

Proposition 1.2.7. *If R is a commutative ring then $R[x_1, \ldots, x_n]$ is a commutative ring.*

Proof. We will prove this by induction on n. Let $n = 1$. Then $R[x_1]$ is a ring by Proposition 1.2.6. Assume that $R[x_1, \ldots, x_{n-1}]$ is a ring for $n > 1$. We want to show that $R[x_1, \ldots, x_n]$ is a ring. By Proposition 1.2.6, we have that $R[x_1, \ldots, x_{n-1}][x_n]$ is a ring. It suffices to show that $R[x_1, \ldots, x_{n-1}][x_n] = R[x_1, \ldots, x_n]$.

Let $s \in R[x_1, \ldots, x_{n-1}][x_n]$. Then $s = \sum_{i=0}^{t} s_i x_n^i$ for $s_i \in R[x_1, \ldots, x_{n-1}]$. Note that if $r \in R[x_1, \ldots, x_{n-1}]$ then $r = \sum_{v_i \geq 0} r_{\underline{v}} x_1^{v_1} x_2^{v_2} \cdots x_{n-1}^{v_{n-1}}$ for $r_{\underline{v}} \in R$ and for all $\underline{v} = (v_1, \ldots, v_{n-1}) \in$

$\mathbb{Z}^{n-1}$. It follows that

$$s = \sum_{i=0}^{t} \left(\sum_{v_j \geq 0} r_{\underline{v}} x_1^{v_1} x_2^{v_2} \cdots x_{n-1}^{v_{n-1}} \right) x_n^i, \tag{1.2.1}$$

for $r_{\underline{v}} \in R$ and for all $\underline{v} \in \mathbb{Z}^{n-1}$. Notice that the i^{th} summand of (1.2.1) is the sum $\sum_{v_j \geq 0} r_{\underline{v}} x_1^{v_1} x_2^{v_2} \cdots x_{n-1}^{v_{n-1}} x_n^i$, for all $\underline{v} \in \mathbb{Z}^{n-1}$. Or, equivalently, the i^{th} summand of (1.2.1) is given by $\sum_{w_j \geq 0} r_{\underline{w}} x_1^{w_1} x_2^{w_2} \cdots x_{n-1}^{w_{n-1}} x_n^{w_n}$ for all $\underline{w} \in \mathbb{Z}^{n-1} \times \{i\}$. It follows that

$$s = \sum_{w_j \geq 0} r_{\underline{w}} x_1^{w_1} x_2^{w_2} \cdots x_{n-1}^{w_{n-1}} x_n^{w_n}$$

for all $\underline{w} \in \mathbb{Z}^{n-1} \times \mathbb{Z} = \mathbb{Z}^n$. Therefore $s \in R[x_1, \ldots, x_n]$. It follows that $R[x_1, \ldots, x_{n-1}][x_n] \subseteq R[x_1, \ldots, x_n]$.

Now let $s \in R[x_1, \ldots, x_n]$. Then

$$s = \sum_{w_j \geq 0} r_{\underline{w}} x_1^{w_1} x_2^{w_2} \cdots x_{n-1}^{w_{n-1}} x_n^{w_n}$$

for all $\underline{w} \in \mathbb{Z}^n$ and for $r_{\underline{w}} \in R$. It follows that

$$s = \sum_{i}^{t} \left(\sum_{v_j \geq 0} r_{\underline{v}} x_1^{v_1} x_2^{v_2} \cdots x_{n-1}^{v_{n-1}} \right) x_n^i$$

for some $t \in \mathbb{Z}$ and for all $\underline{v} \in \mathbb{Z}^{n-1}$. Therefore $s \in R[x_1, \ldots, x_{n-1}][x_n]$. It follows that $R[x_1, \ldots, x_n] \subseteq R[x_1, \ldots, x_{n-1}][x_n]$. □

As a direct result of Proposition 1.2.7, we have Corollary 1.2.8.

Corollary 1.2.8. *$k[x_1, \ldots, x_n]$ is a ring.*

Recall that a group G can have a special kind of subgroup N called a "normal subgroup". Similarly, a ring R can have a special subset I, called an "ideal".

Definition 1.2.9. Let R be a ring, let $I \subseteq R$ be a subset and let $r \in R$. Then I is an *ideal* of R if

1. I is an additive subgroup of R and I is closed under multiplication, and
2. for all $a \in I, r \in R$, we have that $ar, ra \in I$.

$\triangle$

Consider the following example.

Example 1.2.10. Consider the polynomial ring $R = k[x, y]$. Then $I = (x, y^2)$ is a subset of R, where $I = \{r \in R \mid r = sx + ty^2 \text{ for all } s, t \in R\}$, i.e. I is the set of polynomials generated by x and y^2. So, since $x - 3y, 7 + xy \in k[x, y]$, then

$$(x - 3y)x + (7 + xy)y^2 = x^2 - 3xy + 7y^2 + xy^3 \in I \subseteq k[x, y].$$

Observe that I is an additive subgroup of R and that I is closed under multiplication. By definition, I satisfies property (2) from Definition 1.2.9. Thus I is an ideal in R. $\Diamond$

Recall that a group G modulo a normal subgroup N is a quotient group G/N. Similarly, we have that the set R/I, given by a ring R modulo an ideal I, is a "quotient ring" R/I.

Proposition 1.2.11. *Let R be a ring and let I be an ideal of R. Then the additive quotient group R/I is a ring under the binary operations:*

$$(r + I) + (s + I) = (r + s) + I \text{ and } (r + I) \cdot (s + I) = (rs) + I$$

for all $r, s \in R$.

Proof. To prove that R/I is a ring, we need to show that R/I is an abelian group under addition and that it satisfies the properties stated in Definition 1.2.3. Since R is a ring and I is an ideal of R, it follows by definition that R is an additive abelian group and that I is an additive subgroup of R. As subgroups of abelian groups are normal, we have that R/I is an abelian quotient group under addition. Let $x + I, y + I, z + I \in R/I$ be cosets, denoted $\overline{x}, \overline{y}, \overline{z}$, respectively. We define the multiplication of $\overline{x}$ and $\overline{y}$ to be

$$\overline{x} \cdot \overline{y} = (x+I)(y+I) := x \cdot y + I = \overline{xy}.$$[2]

Since $x, y \in R$, it follows that $x \cdot y \in R$. So $x \cdot y + I = \overline{xy} \in R/I$. Thus R/I is closed under multiplication. Using our definition of multiplication of cosets, it follows that $(\overline{x} \cdot \overline{y}) \cdot \overline{z} = \overline{xy} \cdot \overline{z} = (xy+I)(z+I) = xy \cdot z + I = x \cdot yz + I = \overline{x} \cdot \overline{yz}$, so we have associativity in R/I. We also have

$$\begin{aligned}
\overline{x}(\overline{y} + \overline{z}) &= (x+I) \cdot ((y+I) + (z+I)) \\
&= (x+I)((y+z)+I) \\
&= x(y+z) + I \\
&= (xy+xz) + I \\
&= \overline{xy+xz}.
\end{aligned}$$

Hence R/I is distributive over addition. Thus all of the properties in Definition 1.2.3 are satisfied, so R/I is a ring. □

We say that R/I is the *quotient ring* of R by I.

Example 1.2.12. Let $R = k[x,y]$ be a ring and $I = (x, y^2)$ be an ideal in R. Then R modulo its ideal I is the quotient ring $R/I = \dfrac{k[x,y]}{(x,y^2)}$. Recall from Example 1.2.10 that $I \subseteq R$ is made up of ring elements that can each be written as $a = s(x,y)x + t(x,y)y^2 \in I$ for some $s(x,y), t(x,y) \in R$. So, elements of R/I are cosets of the form $r + I$, denoted $\overline{r}$, for $r \in R$. For example, the cosets $\overline{2y} = 2y + I, \overline{3 - xy} = (3 - xy) + I$ are in R/I. Using

[2] See p. 242 in [3] for a proof that this operation is well-defined.

the definitions for addition and multiplication from Proposition 1.2.11, we have that

$$\begin{aligned}\overline{2y} + \overline{3 - xy} &= 2y + I + (3 - xy) + I \\ &= (2y + 3 - xy) + I \in R/I,\end{aligned}$$

and

$$\begin{aligned}\overline{2y} \cdot \overline{3 - xy} &= (2y + I)(3 - xy + I) \\ &= 2y(3 - xy) + I \\ &= (6y - 2xy^2) + I \\ &= \overline{6y - 2xy^2}.\end{aligned}$$

$\Diamond$

Recall that $R = k[x, y]$ is the set of polynomials of the form $\sum_{i=0}^{n} c_i x^{a_i} y^{b_i}$ such that $a_i, b_i, n \in \mathbb{Z} \geq 0, c_i \in k$. Observe that for any polynomial $r \in R$, we can arrange the terms of r in order of degree. Consider $r = 3x^2 + 4xy + 3 - y^3$. Note that $\deg(3x^2) = \deg(4xy) = 2, \deg(3) = 0$, and $\deg(-y^3) = 3$. Let $R_2 \subseteq R$ be the subset of R such that $s_2 \in R_2$ if and only if $\deg(s_2) = 2$ for all $s_2 \in R$. Then $3x^2, 4xy, 3x^2 + 4xy \in R_2$. Similarly, let $R_3 = \{s_3 \in R \mid \deg(s_3) = 3\}$ and let $R_0 = \{s_0 \in R \mid \deg(s_0) = 0\}$. Note that $R_0 = k$. It follows that $3 \in R_0$ and $-y^3 \in R_3$. We can relate r to R_0, R_2, and R_3 using direct sums.

Definition 1.2.13. Let A, B be sets such that $A \cap B = \emptyset$. Then we define

$$A \oplus B := A + B$$

to be the *direct sum* of A and B. $\triangle$

Observe that every $s \in R_0 \oplus R_2 \oplus R_3$ can be written as $s = s_0 + s_2 + s_3$ for some $s_0 \in R_0, s_2 \in R_2, s_3 \in R_3$.

We say $\mathbb{F}\langle r_1, \ldots, r_n\rangle = \{c_1r_1 + c_2r_2 + \ldots + c_nr_n \mid c_i \in \mathbb{F}\}$ is the vector space is the vector space over the field $\mathbb{F}$ spanned by $r_1, \ldots, r_n$. Let I, J be sets. We define the multiplication of these sets as

$$IJ := \{ij \mid i \in I, j \in J\}. \tag{1.2.2}$$

Note that $\deg(s_2s_3) = 5$ for some $s_2 \in R_2, s_3 \in R_3$ and R_2, R_3 are as previously defined. If we define $R_5 = \{s_5 \in R \mid \deg(s_5) = 5\}$, then $s_2s_3 \in R_5$. It follows that $R_2R_3 \subseteq R_5$. By Definition 1.2.14, we find that $R = k[x, y]$ is standard graded.

Definition 1.2.14. Let R be a ring. R is *standard graded* if

1. $R = \bigoplus_{i \geq 0} R_i$, where each R_i is an abelian group over addition and R_0 is a field,
2. $R_iR_j \subseteq R_{i+j}$,
3. R is "generated in degree 1," i.e. $R = R_0[R_1] = R_0[x_1, \ldots, x_n]$ such that $R_1 = R_0\langle x_1, \ldots, x_n\rangle$.

$\triangle$

We say that R_i is the i^{th} graded piece of R. Further, if $r \in R_i$, we say that r is a *homogeneous element of degree* i. Note that each R_i is generated by elements of degree i. Consider the following examples.

Example 1.2.15. Let $R = k[x, y]$ be the polynomial ring in two variables over the field k. Then property 1.2.14(1) is satisfied because

$$R = k \oplus k\langle x, y\rangle \oplus k\langle x^2, xy, y^2\rangle \oplus \ldots.$$

We have that $k\langle x, y\rangle = \{a_1x + a_2y \mid a_i \in k\}$ and $k\langle x^2, xy, y^2\rangle = \{b_1x^2 + b_2xy + b_3y^2 \mid b_i \in k\}$. It is a consequence of (1.2.2) that

$$k\langle x, y\rangle k\langle x^2, xy, y^2\rangle = \{rs \mid r \in k\langle x, y\rangle, s \in k\langle x^2, xy, y^2\rangle.$$

So, $(5x+y)(7x^2-3xy) \in k\langle x,y\rangle k\langle x^2,xy,y^2\rangle$. But we also have that

$$(5x+y)(7x^2-3xy) = 35x^3 - x^2y - 3xy^2 \in k\langle x^3, x^2y, xy^2, y^3\rangle.$$

From our definitions of $k\langle x,y\rangle$ and $k\langle x^2,xy,y^2\rangle$, it follows that

$$k\langle x,y\rangle k\langle x^2,xy,y^2\rangle = \{(a_1x+a_2y)(b_1x^2+b_2xy+b_3y^2) \mid a_i, b_i \in k\}.$$

Equivalently, we have

$$k\langle x,y\rangle k\langle x^2,xy,y^2\rangle = \{a_1b_1x^3 + (a_1b_2+a_2b_1)x^2y + (a_1b_3+a_2b_2)xy^2 + a_2b_3y^3 \mid a_i, b_i \in k\}.$$

Since k is a field, we have that $a_1b_1, a_1b_2+a_2b_1, a_1b_3+a_2b_2, a_2b_3 \in k$. So we can write

$$k\langle x,y\rangle k\langle x^2,xy,y^2\rangle \subseteq \{c_1x^3 + c_2x^2y + c_3xy^2 + cy^3 \mid c_i \in k\},$$

which is, by definition, equivalent to $k\langle x^3, x^2y, xy^2, y^3\rangle$. Thus

$$k\langle x,y\rangle k\langle x^2,xy,y^2\rangle \subseteq k\langle x,y\rangle k\langle x^2,xy,y^2\rangle.$$

Using a similar argument for any two graded pieces of R, we find that R_iR_j is contained in the abelian group with elements of degree $i+j$, so property 1.2.14(2) is satisfied.

Since $R_0 = k$, it follows that $R = R_0[x,y]$. So the final property 1.2.14(3) is also satisfied. $\Diamond$

One of the most important algebraic structures is the module. Modules are similar to vector spaces from linear algebra, but modules can be over any ring, not just a field. As we mention in Example 1.2.17, any module over a field k is a vector space over k.

Definition 1.2.16. Let R be a ring. An *R-module* is a set M together with

1. a binary operation $+$ on M under which M is an abelian group, and

2. an action of R on M: $R \times M \longrightarrow M$, denoted rm for all $r \in R$, $m \in M$. This action satisfies

(a) $(r+s)m = rm + sm$ for all $r, s \in R, m \in M$,

(b) $(rs)m = r(sm)$ for all $r, s \in R, m \in M$,

(c) $r(m+n) = rm + rn$ $r \in R, m, n \in M$, and

(d) $1m = m$ for all $m \in M$, if R has a 1.

△

Example 1.2.17. Let $\mathbb{F}$ be a field and let M be an $\mathbb{F}$-module. We observe that M satisfies all of the properties of a vector space of $\mathbb{F}$. Then M is a vector space over $\mathbb{F}$.

Example 1.2.18. All abelian groups are $\mathbb{Z}$-modules. Let $R = \mathbb{Z}$ and let G be any abelian group under some binary operation, $+$. Then we can define an action of $R = \mathbb{Z}$ on G that satisfies the statements in the previous definition. For any $n \in \mathbb{Z}$ and $g \in G$, we define this action as follows:

$$ng = \begin{cases} g + g + \cdots + g \ (n \text{ times}) & \text{if } n > 0 \\ 0 & \text{if } n = 0 \\ -g - g - \cdots - g \ (-n \text{ times}) & \text{if } n < 0 \end{cases}.$$

This action of $\mathbb{Z}$ on G makes G into a $\mathbb{Z}$-module. ◊

Example 1.2.19. Let $R = F$ be a field. Let $M = F[x]$ be the polynomial ring with one variable over F. Then M is an abelian group under addition. Observe that this is a vector space by Example 1.2.17, so the above properties are satisfied. So M is an R-module. ◊

Now, consider a more complicated example.

Example 1.2.20. Let k be a field and let $R = k[x, y]$ be the polynomial ring with two variables over k. Let $M = \dfrac{k[x,y]}{(x,y^2)}$. Recall from Example 1.2.10 that (x, y^2) is an ideal of $k[x, y]$. We want to show that M is a module over R. Since M is a ring by Example 1.2.12, we have that M is an abelian group under addition. We define $\overline{m} := m + (x, y^2)$ for some

$m \in M$. Observing the action of R on M, we see that $r \cdot \overline{m}$ takes $\overline{m} \in M$ to $\overline{rm} \in M$ for all $r \in R, \overline{m} \in M$. We also have the following:

1. $(r+s)\overline{m} = \overline{rm} + \overline{sm}$ for all $r, s \in R, \overline{m} \in M$,

2. $(rs)\overline{m} = r(\overline{sm})$ for all $r, s \in R, m \in M$, and

3. $r(\overline{m} + \overline{n}) = \overline{rm} + \overline{rn}$ for all $r \in R, \overline{m}, \overline{n} \in M$.

Thus by Definition 1.2.16 M is a module. $\Diamond$

Recall from Definition 1.2.14 that for some graded ring R, $R_i \subseteq R$ is the i^{th} graded piece of R, meaning that every $r \in R_i$ is homogeneous of degree i. Similarly, a graded module M has the subset $M_i \subseteq M$ that is the i^{th} graded piece of M.

Definition 1.2.21. Let R be a ring and let $M = \bigoplus_{i \in \mathbb{Z}} M_i$ be an R-module. If $R_i M_j \subseteq M_{i+j}$, then M is a *graded module*. $\triangle$

Example 1.2.22. Let k be a field and let $R = k[x, y]$. Let $M = (x, y^2)$ be an R-module. Observe that M contains elements of different degree. We can think of M as the direct sum of sets, each containing elements of a different degree:

$$M = (x, y^2) = k\langle x\rangle \oplus k\langle x^2, y^2\rangle \oplus k\langle x^3, xy^2\rangle \oplus k\langle x^4, x^2y^2, y^4\rangle \oplus \cdots .$$

Let $m \in R_i M_j$. Then $m = zw$ for some $z \in R_i$ and $w \in M_j$. So $\deg(z) = i$ and $\deg(w) = j$. It follows that $\deg(m) = i + j$. Therefore $R_i M_j \subseteq M_{i+j}$. It follows from Definition 1.2.21 that M is a graded module. $\Diamond$

Definition 1.2.23 ([3])**.** Let R be a ring. Let A, B be R-modules. A map $f \colon A \longrightarrow B$ is an *R-module homomorphism* if

1. $f(rx) = rf(x)$ for all $x \in A, r \in R$,

2. $f(x + y) = f(x) + f(y)$ for all $x, y \in A$.

An R-module homomorphism is also an R-module *isomorphism* if it is both injective and surjective. Two modules M and N are *isomorphic*, denoted $M \simeq N$ if there is some R-module isomorphism $\phi\colon M \longrightarrow N$. △

When we have an R-module homomorphism from a graded R-module to another graded R-module, the homomorphism may or may not be graded, itself.

Definition 1.2.24 ([10])**.** Let R be a graded ring and let M, N be graded R-modules. Let $f\colon M \longrightarrow N$ be an R-module homomorphism. Then f is *graded of degree* d if $f(M_n) \subseteq N_{n+d}$ for all n. △

Example 1.2.25. Let $R = k$. Consider the homomorphism

$$\phi\colon k[x] \longrightarrow k[x]$$
$$r \mapsto r \cdot x.$$

Notice that for $r \in k[x], a \in k$, we have

$$\phi(ar) = arx = a\phi(r)$$
$$\text{and} \qquad \phi(r+s) = (r+s)x = rx + sx = \phi(r) + \phi(s).$$

Thus ϕ is an R-module homomorphism.

Let $s \in \phi(k\langle x\rangle)$. Then $s = (ax) \cdot x$ for some $a \in k$. So $r = ax^2 \in k\langle x^2\rangle$. Thus $\phi(k\langle x\rangle) \subseteq k\langle x^2\rangle$. Observe that $\phi(k\langle x^i\rangle) \subseteq k\langle x^{i+1}\rangle$ for all $i \geq 1$. Therefore ϕ is graded of degree 1. ♢

We can use "twists" to keep track of the degree shifts of graded R-module homomorphisms.

Definition 1.2.26. Let $S = \bigoplus_{i \geq 0} S_i$ be a standard graded ring. Then $S(n)$ is called the *twist of* S *by* n and is defined by $S(n)_i := S_{i+n}$. △

Example 1.2.27. Consider the polynomial ring $S = k[x, y]$. Recall from Example 1.2.15 that this ring is standard graded, so we can think of S as the direct sum of vector spaces, each of a different degree:

$$S := k[x,y] = k \oplus k\langle x,y\rangle \oplus k\langle x^2, xy, y^2\rangle \oplus \cdots .$$

Note that we can multiply any of the subspaces by a variable and land in a different subspace. For example, multiplying $S_1 = k\langle x,y\rangle$ by xy moves everything in $k\langle x,y\rangle$ to something in $S_3 = k\langle x^3, xy^2, x^2y, y^3\rangle$. We can keep track of these shifts with the twists. That is, for the homomorphism $S \xrightarrow{\cdot xy} S$, we take elements of degree i to elements of degree $i+2$. We would like to have a map that takes degree i elements to degree i (such a map is called *homogeneous*). To do this, we will twist the degrees of the source of the map in order to preserve the degrees, i.e. $S(-2) \xrightarrow{\cdot xy} S$. So, twisting $S_1 = k\langle x,y\rangle$ by 2 gives $S(-2)_3 = S_{3-2} = S_1 = k\langle x,y\rangle$. ♢

Another important structure is the short exact sequence.

Definition 1.2.28. Let R be a ring. A *short exact sequence* is a sequence of two R-module homorphisms f, g between three R-modules A, B, C:

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

such that f is one-to-one, g is onto, and $\operatorname{im}(f) = \ker(g)$. △

Example 1.2.29. Consider the sequence of $\mathbb{Z}$-modules:

$$0 \longrightarrow 2\mathbb{Z} \xrightarrow{f} \mathbb{Z} \xrightarrow{g} \mathbb{Z}/2\mathbb{Z} \longrightarrow 0.$$

Here f is the inclusion map and g takes 1 to $\overline{1}$. Observe that f is a one-to-one map and g is onto. Also note that $\operatorname{im}(f) = 2\mathbb{Z}$ is all of the even integers, all of which g then takes to $\overline{0}$ in $\mathbb{Z}/2\mathbb{Z}$. So $\ker(g) = 2\mathbb{Z}$. Since f is one-to-one and g is onto, it follows that $\operatorname{im}(f) = \ker(g)$. ♢

We will use short exact sequences to construct "resolutions" in Section 1.3.

1.3 Generating sets, free modules, and graded minimal free resolutions

The graded minimal free resolution of a finitely generated R-module is one of the main structures that we will be examining in each of the following sections of this paper. In this section, we will first state the definitions and theorems that are necessary for constructing a graded minimal free resolution. Then we will use these definitions and theorems to describe the construction of a graded minimal free resolution.

Definition 1.3.1. Let R be a ring. Let $M = Rm_1 + Rm_2 + \ldots + Rm_l$ be an R-module such that $\deg(m_i) = d_i$ for all $1 \leq i \leq l$. Then the set $m = \{m_1, m_2, \ldots, m_l\}$ is a *homogeneous generating set of* M. If l is finite, then M is *finitely generated*. $\triangle$

Theorem/Definition 1.3.2 ([12])**.** Let M be a finitely generated graded module over a polynomial ring $R = k[x_1, \ldots, x_n]$. Let $m_1, \ldots, m_n \in M$ be a homogeneous generating set of M and define $\mathfrak{m} = (x_1, \ldots, x_n)$. We say that $m_1, \ldots, m_t$ is a *minimal generating set of* M if $\overline{m_1}, \overline{m_2}, \ldots, \overline{m_t} \in M/\mathfrak{m}M$ is a k-basis of the vector space $M/\mathfrak{m}M$. Notice that $R/\mathfrak{m} \simeq k$.

Remark 1.3.3. We want to show that $M/\mathfrak{m}M$ is a vector space over $k \simeq R/\mathfrak{m}$. Recall from Example 1.2.17 that any module over a field is a also a vector space over that field. So, it suffices to show that $M/\mathfrak{m}M$ is a module over k. Observe that $\mathfrak{m}M$ is, indeed, an ideal of M by Definition 1.2.9. It follows that $M/\mathfrak{m}M$ is an abelian group under $+$. It remains to define an action of k on $M/\mathfrak{m}M$ satisfying the properties in Definition 1.2.16(2). Let $\phi\colon k \times M/\mathfrak{m}M \longrightarrow M/\mathfrak{m}M$ be a map defined by $\phi(a, \overline{m}) = a\overline{m}$. We find that

$$\begin{aligned}(a+b)\overline{m} &= (a+b)(m+\mathfrak{m}M) = a(m+\mathfrak{m}M) + b(m+\mathfrak{m}M) = (am+\mathfrak{m}M) + (bm+\mathfrak{m}M)\\ &= \overline{am} + \overline{bm},\\ (ab)\overline{m} &= ab(m+\mathfrak{m}M) = a(bm+\mathfrak{m}M) = a\overline{bm},\end{aligned}$$

and

$$a(\overline{m} + \overline{n}) = a(m + \mathfrak{m}M + n + \mathfrak{m}M) = a(m + \mathfrak{m}M) + a(n + \mathfrak{m}M)$$
$$= (am + \mathfrak{m}M) + (an + \mathfrak{m}M) = \overline{am} + \overline{an}.$$

Definition 1.3.4 ([3])**.** Let R be a ring. An R-module F is said to be *free* on the subset A of F if for every nonzero element x of F, there exist unique nonzero elements $r_1, r_2, \ldots, r_n$ of R and unique $a_1, a_2, \ldots, a_n$ in A such that $x = r_1a_1 + r_2a_2 + \cdots + r_na_n$, for some $n \in \mathbb{Z}^+$. △

Theorem 1.3.5 ([9])**.** *Let R be a graded ring and let M be a finitely generated graded R-module. Then M is the homomorphic image of a graded free R-module. In other words, there exists a graded free R-module F and a surjective graded R-module homomorphism,*

$$\pi\colon F = \bigoplus_{i=1}^{t} R(n_i) \longrightarrow M = Rm_1 + \cdots + Rm_t.$$

Proof. Let $M = Rm_1 + Rm_2 + \cdots + Rm_t$ be a finitely generated graded R-module such that $\deg(m_i) = d_i$. Let $F = \bigoplus_{i=1}^{t} R(-d_i)$ be a graded free R-module. Consider the map

$$\phi\colon F \longrightarrow M,$$
$$(r_1, r_2, \cdots, r_t) \mapsto r_1m_1 + r_2m_2 + \cdots + r_tm_t.$$

We want to show that ϕ is a graded R-module homomorphism and that ϕ is surjective. Let $(x_1, \ldots, x_t), (y_1, \ldots, y_t) \in F$ and let $s \in R$. Then

$$\phi(x_1, x_2, \ldots, x_t) = x_1m_1 + x_2m_2 + \cdots + x_tm_t,$$

and

$$\phi(y_1, y_2, \ldots, y_t) = y_1m_1 + y_2m_2 + \cdots + y_tm_t.$$

We have that

$$\begin{aligned}\phi(s(x_1,x_2,\ldots,x_t)) &= \phi((sx_1,sx_2,\ldots,sx_t))\\ &= sx_1m_1+sx_2m_2+\cdots+sx_tm_t\\ &= s(x_1m_1+x_2m_2+\cdots+x_tm_t)\\ &= s\phi(x_1,x_2,\ldots,x_t).\end{aligned}$$

We also have that

$$\begin{aligned}\phi(r_1,\ldots,r_t)+\phi(s_1,\ldots,s_t) &= r_1m_1+\cdots+r_tm_t+s_1m_1+\cdots+s_tm_t\\ &= (r_1+s_1)m_1+\cdots+(r_t+s_t)m_t\\ &= \phi(r_1+s_1,r_2+s_2,\ldots,r_t+s_t)\\ &= \phi((r_1,\ldots,r_t)+(s_1,\ldots,s_t)).\end{aligned}$$

Thus ϕ is an R-module homomorphism by Definition 1.2.23.

Now we need to check that ϕ is graded. Consider the n^{th} graded piece of F, given by $F_n = \bigoplus_{i=1}^{t} R(-d_i)_n$. Let $(s_1,\ldots,s_t)\in F_n$ such that $\deg(s_i)=n-d_i$ for all $1\le i\le n$. Then $\phi(s_1,\ldots,s_t)=s_1m_1+\cdots+s_tm_t$, where $\deg(m_i)=d_i$ for all $1\le i\le n$. So, for the i^{th} summand in $s_1m_1+\cdots+s_tm_t$, it follows that $\deg(s_im_i)=n-d_i+d_i=n$. Therefore $\deg(s_1m_1+\cdots+s_tm_t)=n$. It follows that $\phi(s_1,\ldots,s_t)\in M_n$. Thus we have that $\phi(F_n)\subseteq M_n$. It follows from Definition 1.2.24 that ϕ is graded of degree 0.

It remains to show that ϕ is surjective. Let $g\in M$. Then $g=\sum_{i=1}^{t} a_im_i$ for some $a_i\in R$. From the definition of ϕ, we have that $\phi(0,\ldots,0,a_i,0,\ldots,0)=a_im_i$, where a_i is in the i^{th} place in the t-tuple and the rest of the entries are 0. Then

$$g=\phi(a_1,0,\ldots,0)+\phi(0,a_2,\ldots,0)+\phi(0,0,a_3,\ldots,0)+\cdots+\phi(0,\ldots,0,a_t),$$

where each t-tuple has a_i in the i^{th} place and zeros in the remaining entries. Since we have already shown that ϕ is an R-module homomorphism, it follows that

$$g = \phi((a_1, 0, \ldots, 0) + (0, a_2, \ldots, 0) + (0, 0, a_3, \ldots, 0) + \cdots + (0, \ldots, 0, a_t))$$
$$= \phi(a_1, a_2, \ldots, a_t).$$

Thus we have that ϕ is surjective. □

Proposition 1.3.6. *Let R be a ring. Then M is a cyclic R-module if and only if there is an ideal $I \subseteq R$ such that $M \simeq R/I$.*

Proof. First we will prove that if $M \simeq R/I$ then M is a cyclic R-module. Suppose that $M \simeq R/I$ for some ideal $I \subseteq R$. Then the map $\rho\colon R/I \longrightarrow M$ is an isomorphism. Let $\rho(\overline{1}) = n$ for some $n \in M$. We want to show that $M = R \cdot n$. It suffices to show that for all $m \in M$ we have that $m = r \cdot n$. Let $m \in M$. Since ρ is an isomorphism, then ρ is onto. It follows that there is some $\overline{r} \in R/I$ such that $\rho(\overline{r}) = m$. Then since ρ is a homomorphism, we have that $\rho(\overline{r}) = \rho(r \cdot \overline{1}) = r\rho(\overline{1}) = r \cdot n$.

Next we will prove that if M is cyclic then $M \simeq R/I$ for some ideal $I \subseteq R$. Suppose that M is a cyclic R-module. Then $M = R \cdot n$ for some $n \in M$. We want to show that there is some ideal $I \subseteq R$ such that $M \simeq R/I$. Using Theorem 1.3.5, we have the surjection $\pi\colon R \longrightarrow M$ defined by $\pi(r) = r \cdot n$ for some $r \in R$, $n \in M$. Consider the short exact sequence,

$$0 \longrightarrow K \longrightarrow R \xrightarrow{\pi} M \longrightarrow 0,$$

where $K = \ker(\pi) = \{r \in R \mid r \cdot n = 0\}$. It follows from the First Isomorphism Theorem of modules [3, Theorem 10.2.4(1)] that $M \simeq R/K$. □

Definition 1.3.7. Let M be a graded R-module. A *graded free resolution* $F_\bullet$ of M is an exact sequence of graded R-modules and graded R-module homomorphisms

$$F_\bullet\colon \qquad 0 \longleftarrow M \xleftarrow{\pi_0} F_0 \xleftarrow{\rho_0} F_1 \xleftarrow{\rho_1} F_2 \longleftarrow \cdots.$$

△

Theorem 1.3.8 ([4, Theorem 1.13 (Hilbert Syzygy Theorem)]). *If $R = k[x_1, \ldots, x_r]$, then every finitely generated graded R-module has a finite graded free resolution of length $\leq r$, by finitely generated free modules.*

We construct the unique graded minimal free resolution of M using free modules and minimal generating sets.

Remark 1.3.9. Let $R = k[x_1, \ldots, x_n]$ and let M be a finitely generated graded R-module. The unique graded *minimal free resolution* of M is constructed using minimal generating sets. Let $\{m_1, \ldots, m_t\}$ be a minimal generating set of M. Then it follows from Definition 1.3.1 that $M = Rm_1 + Rm_2 + \cdots + Rm_t$, where $\deg(m_i) = d_i$ for all $1 \leq i \leq t$. By Theorem 1.3.5, we have the surjective graded R-module homomorphism

$$\pi_0 \colon F_0 = \bigoplus_{i=1}^{t} R(d_i) \longrightarrow M$$

defined by $\pi_0(r_1, r_2, \ldots, r_t) = r_1 m_1 + r_2 m_2 + \cdots + r_t m_t$. Let $K_0 = \ker(\pi_0)$. Since M is finitely generated, we note that F_0 is also finitely generated, by construction. Consider the short exact sequence

$$0 \longrightarrow K_0 \xrightarrow{\phi_0} F_0 \xrightarrow{\pi_0} M \longrightarrow 0. \tag{1.3.1}$$

It follows that $K_0 = \mathrm{im}(\phi_0) \subseteq F_0$. Since F_0 is finitely generated, it follows from Remark 1.1.1 that K_0 is finitely generated. Let $\{s_1, \ldots, s_l\}$ be a minimal generating set of K_0 such that $\deg(s_i) = e_i$. Then by Theorem 1.3.5 we have the surjective graded R-module homomorphism

$$\pi_1 \colon F_1 = \bigoplus_{i=1}^{l} R(e_i) \longrightarrow K_0.$$

Let $\sigma_1 \colon F_1 \longrightarrow F_0$ be the map defined by $\sigma_1(f) = \phi_0(\pi_1(f))$. Note that F_1 is finitely generated by construction. Then we have

$$\begin{tikzcd}
0 & M \arrow[l] & F_0 \arrow[l, "\pi_0"'] & & F_1. \arrow[ll, "\sigma_0"'] \arrow[dl, "\pi_1"'] \\
& & & K_0 \arrow[ul, "\phi_0"] \arrow[dl] & \\
& & 0 & & 0 \arrow[ul]
\end{tikzcd}$$

Let $K_1 = \ker(\pi_1)$. Then we have the short exact sequence

$$0 \longrightarrow K_1 \xrightarrow{\phi_1} F_1 \xrightarrow{\pi_1} K_0 \longrightarrow 0.$$

It follows that $K_1 \subseteq F_1$. Since F_1 is finitely generated, it follows from Remark 1.1.1 that K_1 is finitely generated. Then by 1.3.5 there is a surjective graded R-module homomorphism $\pi_2\colon F_2 \longrightarrow K_1$. Let $\sigma_1\colon F_2 \longrightarrow F_1$ be the map defined by $\sigma_1(f) = \phi_1(\pi_2(f))$. Then we have

$$\begin{tikzcd}
0 & M \arrow[l] & F_0 \arrow[l, "\pi_0"'] & F_1 \arrow[l, "\sigma_0"'] & & F_2. \arrow[ll, "\sigma_1"'] \arrow[dl, "\pi_2"'] \\
& & & & K_1 \arrow[ul, "\phi_1"] \arrow[dl] & \\
& & & 0 & & 0 \arrow[ul]
\end{tikzcd}$$

Let $K_2 = \ker(\pi_2)$. We can continue in this manner until we are left with $\ker(\pi_r) = 0$ for some $1 \leq r \leq n$ (given by Theorem 1.3.8). The result is the graded minimal free resolution of M:

$$0 \longleftarrow M \xleftarrow{\sigma_0} F_0 \xleftarrow{\sigma_1} F_1 \xleftarrow{\sigma_2} \cdots \xleftarrow{\sigma_r} F_r \longleftarrow 0.$$

$\diamondsuit$

Example 1.3.10. Consider the ring $R = k[x, y]$ and the R-module $M = \dfrac{k[x, y]}{(x, y^2)}$. By Theorem 1.3.5, there exists a surjective graded R-module homomorphism from some free R-module onto M. Note that M is R modulo an ideal, so by Proposition 1.3.6 it follows that M is cyclic. Therefore the minimal number of generators of M is 1. So, to construct the minimal free resolution of $\dfrac{k[x, y]}{(x, y^2)}$ as an R-module, we start with the natural map $\sigma_0\colon k[x, y] \longrightarrow \dfrac{k[x, y]}{(x, y^2)}$ that takes 1 to $\overline{1}$:

$$0 \longleftarrow \frac{k[x, y]}{(x, y^2)} \xleftarrow{\sigma_0} k[x, y].$$

Then the kernel of σ_0 is (x, y^2), and the inclusion map $\phi_0 \colon (x, y^2) \longrightarrow k[x, y]$ is one-to-one, giving the short exact sequence:

$$0 \longrightarrow (x, y^2) \xrightarrow{\phi_0} k[x, y] \xrightarrow{\sigma_0} \frac{k[x, y]}{(x, y^2)} \longrightarrow 0.$$

Note that (x, y^2) is finitely generated. Then by Theorem 1.3.5, we have a graded surjective R-module homomorphism

$$\sigma_1 \colon k[x, y](-1) \oplus k[x, y](-2) \longrightarrow (x, y^2)$$

defined by $\sigma_1\left(\begin{bmatrix} r \\ s \end{bmatrix}\right) = rx + sy^2$. The map σ_1 composed with ϕ_0 gives us a map

$$\begin{aligned} \rho_0 = \phi_0 \circ \sigma_1 \colon \begin{array}{c} k[x, y](-1) \\ \oplus \\ k[x, y](-2) \end{array} &\longrightarrow k[x, y] \\ \begin{bmatrix} r \\ s \end{bmatrix} &\mapsto rx + sy^2. \end{aligned}$$

Then we have

$$\begin{tikzcd}
0 & \frac{k[x,y]}{(x,y^2)} \arrow[l] & k[x,y] \arrow[l, "\sigma_0"'] & & \begin{array}{c} k[x, y](-1) \\ \oplus \\ k[x, y](-2) \end{array} \arrow[ll, "\rho_0"'] \arrow[dl, "\sigma_1"] \\
& & & (x, y^2) \arrow[ul, "\phi_0"] \arrow[dl] & \\
& & 0 & & 0 \arrow[ul]
\end{tikzcd} .$$

It follows that

$$\ker\sigma_1 = \left\{ \begin{bmatrix} r \\ s \end{bmatrix} \mid rx + sy^2 = 0 \right\} = \begin{array}{c} -y^2 k[x, y] \\ \oplus \\ xk[x, y] \end{array} .$$

We define the inclusion map

$$\phi_1 \colon \begin{array}{c} -y^2 k[x, y] \\ \oplus \\ xk[x, y] \end{array} \longrightarrow \begin{array}{c} k[x, y](-1) \\ \oplus \\ k[x, y](-2) \end{array} .$$

Since $\begin{array}{c}-y^2k[x,y]\\ \oplus\\ xk[x,y]\end{array}$ is finitely generated, then by Theorem 1.3.5 we have the graded surjective R-module homomorphism

$$\sigma_2\colon k[x,y](-3) \longrightarrow \begin{array}{c}-y^2k[x,y]\\ \oplus\\ xk[x,y]\end{array}$$

$$\sigma_2(r) \mapsto \begin{bmatrix} -ry^2 \\ rx \end{bmatrix}.$$

Let $\rho_1 = \phi_1 \circ \sigma_2\colon k[x,y](-3) \longrightarrow \begin{array}{c}-y^2k[x,y]\\ \oplus\\ xk[x,y]\end{array}$. Then we have

$$\begin{tikzcd}
0 & \frac{k[x,y]}{(x,y^2)} \arrow[l] & k[x,y] \arrow[l, "\sigma_0"'] & \begin{array}{c}k[x,y](-1)\\ \oplus\\ k[x,y](-2)\end{array} \arrow[l, "\rho_0"'] & & k[x,y](-3) \arrow[ll, "\rho_1"'] \arrow[dl, "\sigma_2"] \\
& & & & \begin{array}{c}-y^2k[x,y](-1)\\ \oplus\\ xk[x,y](-2)\end{array} \arrow[ul, "\phi_1"] \arrow[dl] & \\
& & & 0 & & 0 \arrow[ul]
\end{tikzcd}$$

Then $\ker\sigma_2 = \left\{ r \in k[x,y] \mid \begin{bmatrix} -ry^2 \\ rx \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\} = 0$. Therefore the graded minimal free resolution of $\frac{k[x,y]}{(x,y^2)}$ is

$$0 \longleftarrow \frac{k[x,y]}{(x,y^2)} \xleftarrow{\sigma_0} k[x,y] \xleftarrow{\rho_0} \begin{array}{c}k[x,y](-1)\\ \oplus\\ k[x,y](-2)\end{array} \xleftarrow{\rho_1} k[x,y](-3) \longleftarrow 0.$$

$\diamondsuit$

1.4 Betti diagrams and Betti decomposition

We can record information about the twists and generators of any resolution with its unique Betti diagram. Betti diagrams require a very specific indexing to keep track of this information.

Definition 1.4.1. Let $V := \bigoplus_{i=0}^{n} \bigoplus_{j\in\mathbb{Z}} \mathbb{Q}$. A *diagram* is an array $D \in V$. △

By convention, we use dashes in place of zeros, and do not distinguish between diagrams with the same non-zero entries. For example, let $V = \bigoplus_{i=0} \bigoplus_{\mathbb{Z}} \mathbb{Q}$. Then

$$D = \begin{pmatrix} \vdots & \vdots & \vdots \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \end{pmatrix} = \begin{pmatrix} 1 & - & - \\ - & 1 & 1 \end{pmatrix} \in V$$

is a diagram.

A "Betti diagram" is a diagram with "Betti numbers" as entries. These Betti numbers are given by the degrees of the different graded pieces of a graded free resolution.

Definition 1.4.2. Let S be a standard graded ring. Let M be a graded free S-module with free resolution

$$F_\bullet: \qquad 0 \longleftarrow M \xleftarrow{\rho_0} F_0 \xleftarrow{\rho_1} F_1 \xleftarrow{\rho_2} F_2 \longleftarrow \cdots,$$

Each F_i is a direct sum of graded pieces,

$$F_i = \bigoplus S(-j)^{\beta_{ij}}$$

where β_{ij} is the number of summands $S(-j)$ in F_i. We call β_{ij} the ij^{th} *Betti number of* M and denote it $\beta_{ij}(M)$. The *Betti diagram* of M is given by

$$\beta(M) = \begin{pmatrix} \beta_{0,0} & \beta_{1,1} & \beta_{2,2} & \cdots & \beta_{n,n} \\ \beta_{0,1} & \beta_{1,2} & \beta_{2,3} & \cdots & \beta_{n,n+1} \\ \vdots & & & \ddots & \end{pmatrix}.$$

△

Or, equivalently, $\beta_{ij}(M)$ is the number of degree j generators of a basis of F_i. Let's construct the Betti diagram of the graded module $M = \dfrac{k[x,y]}{(x,y^2)}$. Recall the resolution $F_\bullet$

of $\dfrac{k[x,y]}{(x,y^2)}$:

$$0 \longleftarrow \frac{k[x,y]}{(x,y^2)} \longleftarrow k[x,y] \longleftarrow \quad k[x,y](-1) \oplus k[x,y](-2) \quad \longleftarrow k[x,y](-3) \longleftarrow 0,$$

where $F_0 = k[x,y], F_1 = k[x,y](-1) \oplus k[x,y](-2), F_2 = k[x,y](-3)$. The entries in column 0 of the Betti diagram $\beta(M)$ are given by $F_0 = \bigoplus k[x,y](-j)^{\beta_{0j}(M)}$. Since $F_0 = k[x,y] = k[x,y](0)^1$, then $\beta_{00}(M) = 1$ is the only one non-zero entry in column 0. The entries in column 1 of $\beta(M)$ are given by

$$\begin{aligned} F_1 &= \bigoplus k[x,y](-j)^{\beta_{1,j}(M)} \\ &= k[x,y](-1)^1 \oplus k[x,y](-2)^1, \end{aligned}$$

so $\beta_{11}(M) = 1$ and $\beta_{12}(M) = 1$ are in column 1 of $\beta(M)$. Similarly, we find that $\beta_{22}(M) = 0$ and $\beta_{23}(M) = 1$ are in column 2 of $\beta(M)$. So, the complete Betti diagram of $\dfrac{k[x,y]}{(x,y^2)}$ is

$$\beta\left(\frac{k[x,y]}{(x,y^2)}\right) = \begin{pmatrix} 1 & 1 & - \\ - & 1 & 1 \end{pmatrix}.$$

As mentioned in Section 1.1, one of Boij and Söderberg's first conjecture was that Betti diagrams could be written as linear combination of pure diagrams (See Theorem 1.4.4). Eisenbud and Schreyer later developed an algorithm that decomposes Betti diagrams into linear combinations of pure diagrams. This algorithm requires "degree sequences" to keep track of the information stored in a Betti diagram as it is decomposed.

We call a diagram A a *pure diagram* if it has at most one entry in each column. For example, the following diagram is a pure diagram:

$$A = \begin{pmatrix} 1 & 2 & \frac{2}{3} & - \\ - & - & - & 6 \end{pmatrix}. \tag{1.4.1}$$

Definition 1.4.3. The n-tuple $d \in \mathbb{Z}^n$ is a *degree sequence* if $d_k < d_{k+1}$ for all k. We can compare two degree sequences: $d \leq d'$ if $d_i \leq d'_i$ for all i. △

Each pure diagram A has a corresponding "degree sequence" $d \in \mathbb{Z}^n$ that tells us the form of A, i.e. where the nonzero entries lie in A. The indexing of A is constructed in such a way that the nonzero entry of A in column r and row s is given by $a_{r,r+s} = a_{ij}$. Each nonzero entry a_{ij} corresponds to $d_i = j$ in the degree sequence of A. Before stating a formal definition, let's work through an example of a pure diagram and its degree sequence.

Consider the pure diagram 1.4.1. The nonzero entries in A are $a_{00} = 1, a_{11} = 2, a_{22} = \frac{2}{3}$, and $a_{34} = 6$. The degree sequence only holds information about where the nonzero entries are in A and does not care what the values actually are. To construct the degree sequence of A, we only need the list of the i, j ordered pairs corresponding to the locations of nonzero entries in A. To construct the degree sequence d using this list, we let $d_i = j$. So $d_0 = 0, d_1 = 1, d_2 = 2, d_3 = 4$, and the degree sequence of A is $d = (0, 1, 2, 4)$.

The location of nonzero Betti numbers of a Betti diagram is important because it holds information about the degree shifts of the corresponding minimal free resolution. Let S be a ring and let M be an S-module. Let $F_\bullet$ be the minimal free resolution of M. Recall from Definition 1.4.2 that the ij^{th} Betti number of M, denoted β_{ij}, is the number of copies of S twisted by j in F_i. So, the location of Betti numbers in the Betti diagram $\beta(M)$ is directly related to the degree shifts in $F_\bullet$.

The conjecture of Boij and Söderberg view the diagrams as just sitting inside some vector space, so we should be able to write them as linear combinations of basis elements. This conjecture, stated as follows in Theorem 1.4.4, was later proved by Eisenbud and Schreyer.

Theorem 1.4.4 ([2],[5])**.** *Let M be a module of finite length. Then there is a unique chain of degree sequences $\{d_0 \leq \cdots \leq d_s\}$ and unique scalars $a_i \in \mathbb{Q}$ such that*

$$\beta(M) = \sum_{i=0}^{s} a_i \pi(d_i),$$

where each $\pi(d_i)$ is a pure diagram.

An algorithm was developed by Eisenbud and Schreyer to figure out this linear combination.

Algorithm 1.4.5 ([5])**.** Let β be a diagram.

1. Find the top-most nonzero entry in each column of β and construct the degree sequence $d = (a_0, a_1, a_2, \ldots, a_n)$ corresponding to these top-most nonzero entries. Construct an elimination matrix $\pi(d)$ from this degree sequence d:
$$\pi_{ij}(d) = \begin{cases} 0 & \text{if } j \neq d_i \\ \Pi_{i\neq i'} \dfrac{1}{|d_i - d_{i'}|} & \text{if } j = d_i \end{cases}.$$

2. Find a maximal $k \in \mathbb{Z}^+$ such that each entry of $\beta - k\pi$ is greater than or equal to 0.

3. Go back to step 1 with $\beta - k\pi$ instead of β. Repeat until each entry of the matrix in step 2 is exactly 0.

Example 1.4.6. We compute the Betti decomposition of the diagram
$$\beta = \begin{pmatrix} 1 & 1 & - \\ - & 1 & 1 \end{pmatrix}.$$

Given this diagram, we want to decompose β down to a linear combination of pure diagrams. Following the algorithm, we find and circle the top-most nonzero entries in each column:

$$\beta = \begin{pmatrix} 1 & 1 & - \\ - & 1 & 1 \end{pmatrix}.$$

The form of these circled entries give the degree sequence $d_0 = (0, 1, 3)$, which gives us the elimination matrix $\pi(d_0) = \begin{pmatrix} \frac{1}{3} & \frac{1}{2} & - \\ - & - & \frac{1}{6} \end{pmatrix}$.

Now we need to find the largest positive integer k such that each entry in $\beta - k\pi(d_0)$ is greater than or equal to 0. In particular, we need the largest integer solution to the inequalities

$$1 - \frac{k}{3} \geq 0,$$
$$1 - \frac{k}{2} \geq 0,$$
$$1 - \frac{k}{6} \geq 0.$$

From these inequalities, we find that k is at most 2. So, we subtract $2\pi(d_0)$ from β:

$$\beta - 2\pi(d_0) = \begin{pmatrix} \frac{1}{3} & - & - \\ - & 1 & \frac{2}{3} \end{pmatrix}.$$

Starting again with step 1, we circle the top-most entries of $\beta - 2\pi(d_0)$:

$$\begin{pmatrix} 1/3 & - & - \\ - & 1 & 2/3 \end{pmatrix}$$

and construct the corresponding degree sequence $d_1 = (0, 2, 3)$ and matrix

$$\pi(d_1) = \begin{pmatrix} \frac{1}{6} & - & - \\ - & \frac{1}{2} & \frac{1}{3} \end{pmatrix}.$$

Observe that $2\pi(d_1) = \beta - 2\pi(d_0)$. Then $\beta - 2\pi(d_0) - 2\pi(d_1) = 0$. So, we are left with $\beta = 2\pi(d_0) + 2\pi(d_1)$. $\Diamond$

So now we have this nice algorithm that lets us write diagrams as linear combinations of pure diagrams (or basis elements). However, this Betti decomposition algorithm does not work nicely for any and all diagrams.

1.5 Hilbert function

Given a diagram D, how can we tell whether or not it is a Betti diagram? Right away, we can figure out what the corresponding resolution would look like based on the entries in D. For example, consider the diagram

$$D = \begin{pmatrix} 1 & - & 1 \\ - & - & - \end{pmatrix}.$$

The corresponding resolution would look like this:

$$0 \longleftarrow M \longleftarrow S \longleftarrow 0 \longleftarrow S(-2) \longleftarrow 0,$$

and there is no M for which this is a resolution. Therefore D is not a Betti diagram. But there are more complicated diagrams that we cannot immediately dismiss as not being a Betti diagram. One way to check whether a module M exists for a given resolution is to see whether the dimensions match up. We can do this using the Hilbert function.

Definition 1.5.1. The *Hilbert function* gives us the vector space dimension of a module M over a field k with respect to the graded degree l. The function is given by

$$H_M(l) := \dim_k M_l.$$

$\triangle$

Lemma 1.5.2. *Let $S = k[x_1, \ldots, x_n]$ be a ring. Then $dim_k S_l = \binom{l+n-1}{n-1}$ is the number of degree l monomials in S.*

Proof. We use the stars and bars technique, where each arrangement of stars and bars represents a degree l monomial in S. In this case, the stars are the variables making up the monomial and the bars separate the stars in terms of variable type. We find that there are $n-1$ bars and l stars, so there are $l+n-1$ total places for stars and bars. We first choose the positions of the $n-1$ bars, which forces the positions of the stars. It follows that there are $\binom{l+n-1}{n-1}$ possible arrangements of stars and bars. $\square$

Example 1.5.3. Let $S = k[x, y, z]$. Suppose we want to figure out how many monomials of degree 4 are in S. Note that each degree 4 monomial in S is consists of 4 variables, which are some combination of x, y, and z. We can represent this using stars for the variables in each monomial and bars for the 3 different degree 1 variables in S. We will use two bars to separate the four stars into three sections, which correspond to the three degree 1 variables in S. For example

$$|{*}{*}{*}|{*}$$

represents the degree 4 monomial y^3z. In this way, each degree 4 monomial in S can be represented by a different configuration of 4 stars and 2 bars. To count the number of degree four monomials in S, we need to count the number of possible configurations of four stars and two bars. Observe that there are 6 objects total that we can rearrange, and that choosing where to place the bars will force the positions of the stars. So, there are $\binom{6}{2} = 15$ degree 4 monomials in S:

$$\begin{array}{rlrlrlrlrl}
z^4: & ||{*}{*}{*}{*} & yz^3: & |{*}|{*}{*}{*} & y^2z^2: & |{*}{*}|{*}{*} & y^3z: & |{*}{*}{*}|{*} & y^4: & |{*}{*}{*}{*}| \\
xz^3: & {*}||{*}{*}{*} & xyz^2: & {*}|{*}|{*}{*} & xy^2z: & {*}|{*}{*}|{*} & xy^3: & {*}|{*}{*}{*}| & & \\
x^2z^2: & {*}{*}||{*}{*} & x^2yz: & {*}{*}|{*}|{*} & x^2y^2: & {*}{*}|{*}{*}| & & & & \\
x^3z: & {*}{*}{*}||{*} & x^3y: & {*}{*}{*}|{*}| & & & & & & \\
x^4: & {*}{*}{*}{*}|| & & & & & & & &
\end{array}.$$

$\diamondsuit$

It follows that we can express the number of degree l monomials in S as $\binom{l+n-1}{n-1}$ or, equivalently $\binom{l+n-1}{l}$. Lemma 1.5.2 along with a generalized version of the rank-nullity theorem, stated in Fact 1.5.4, allows us to compute the Hilbert function.

Fact 1.5.4 ([7, Theorem 5.3.8])**.** Let

$$0 \longrightarrow K \longrightarrow A \xrightarrow{f} C \longrightarrow 0$$

be a short exact sequence of k-vector spaces. Then

$$\dim_k(\ker(f)) + \dim_k(\mathrm{im}(f)) = \dim_k(A).$$

Note that since the sequence in Fact 1.5.4 is exact, we have that $\ker(f) = A$ and $\mathrm{im}(f) = C$. It follows that for any short exact sequence of vector spaces over k

$$0 \longrightarrow K \longrightarrow A \longrightarrow C \longrightarrow 0,$$

we have that $\dim_k(A) = \dim_k(K) + \dim_k(C)$.

Proposition 1.5.5. *If $F_\bullet$ is an exact sequence of k-vector spaces,*

$$F_\bullet\colon \qquad 0 \longleftarrow M \xleftarrow{\rho_0} F_0 \xleftarrow{\rho_1} F_1 \xleftarrow{\rho_2} F_2 \longleftarrow \cdots \longleftarrow F_n \longleftarrow 0,$$

then $\dim_k M = \sum_{i=0}^{n} (-1)^i \dim_k F_i$.

Proof. Let $F_\bullet$ be an exact sequence of k-vector spaces,

$$F_\bullet\colon \qquad 0 \longleftarrow M \xleftarrow{\rho_0} F_0 \xleftarrow{\rho_1} F_1 \xleftarrow{\rho_2} F_2 \xleftarrow{\rho_3} \cdots \xleftarrow{\rho_n} F_n \longleftarrow 0.$$

We will use the notation $K_i = \ker(\rho_i)$ for all $0 \leq i \leq n$. Consider the kernel of ρ_0, denoted K_0. Then we have the short exact sequence

$$0 \longrightarrow K_0 \longrightarrow F_0 \xrightarrow{\rho_0} M \longrightarrow 0.$$

It follows from Fact 1.5.4 that $\dim_k F_0 = \dim_k K_0 + \dim_k M$. So

$$\dim_k M = \dim_k F_0 - \dim_k K_0. \tag{1.5.1}$$

Since $F_\bullet$ is an exact sequence, we have that $\ker(\rho_i) = \operatorname{im}(\rho_{i+1})$ for all $0 \leq i \leq n$. It follows that $K_0 = \operatorname{im}(\rho_1)$. Then $K_0 \subseteq F_0$. So we have the inclusion map $\sigma_0\colon K_0 \longrightarrow F_0$. We also have the surjection $\rho_1\colon F_1 \longrightarrow K_0$. So

$$\begin{tikzcd}
F_\bullet\colon & 0 & M \arrow[l] & & F_0 \arrow[ll, "\rho_0"'] & & F_1 \arrow[ll, "\rho_1"'] \arrow[dl, "\rho_1"] & \cdots \arrow[l, "\rho_2"'] & F_n \arrow[l, "\rho_n"'] & 0 \arrow[l] . \\
& & & & & K_0 \arrow[ul, "\sigma_0"] \arrow[dl] & & & & \\
& & & & 0 & & 0 \arrow[ul] & & &
\end{tikzcd} \tag{1.5.2}$$

Now consider K_1. Since $F_\bullet$ is exact, we have that $K_1 = \operatorname{im}(\rho_2)$. Then $K_1 \subseteq F_1$. So we have the inclusion map $\sigma_1\colon K_1 \longrightarrow F_1$ and the surjection $\rho_2\colon F_2 \longrightarrow K_1$. Adding these maps to

(1.5.2), we have

$$\begin{tikzcd}
\cdots & & F_0 \arrow[ll, "\rho_0"'] & & F_1 \arrow[ll, "\rho_1"'] \arrow[dl, "\rho_1"] & & F_2 \arrow[ll, "\rho_2"'] \arrow[dl, "\rho_2"] & & \cdots \arrow[ll, "\rho_3"'] \\
& & & K_0 \arrow[ul, "\sigma_0"] \arrow[dl] & & K_1 \arrow[ul, "\sigma_1"] \arrow[dl] & & & . \\
& & 0 & & 0 \arrow[ul] & & 0 \arrow[ul] & &
\end{tikzcd}$$

Observe that we now have the short exact sequence

$$0 \longrightarrow K_1 \xrightarrow{\sigma_1} F_1 \xrightarrow{\rho_1} K_0 \longrightarrow 0.$$

It follows from Fact 1.5.4 that $\dim_k F_1 = \dim_k K_1 + \dim_k K_0$. Then $\dim_k K_0 = \dim_k F_1 - \dim_k K_1$. Substituting $\dim_k F_1 - \dim_k K_1$ for $\dim_k K_0$ in (1.5.1), we have that

$$\begin{aligned}
\dim_k M &= \dim_k F_0 - (\dim_k F_1 - \dim_k K_1) \\
&= \dim_k F_0 - \dim_k F_1 + \dim_k K_1.
\end{aligned}$$

Using a similar argument, we find that $\dim_k K_1 = \dim_k F_2 - \dim_k K_2$. It follows that

$$\dim_k M = \dim_k F_0 - \dim_k F_1 + \dim_k F_2 - \dim_k K_2.$$

In fact, we find that $\dim_k K_i = \dim_k F_{i+1} - \dim_k K_{i+1}$ for all $1 \le i \le n$. It follows that $\dim_k M = \displaystyle\sum_{i=0}^{n} (-1)^i \dim_k F_i$. □

Example 1.5.6. Let S be the polynomial ring $k[x, y]$. Consider the resolution of $M = k[x, y]/(x, y^2) = S/(x, y^2)$ and its corresponding Betti diagram $\beta(M)$:

$$0 \longleftarrow M \longleftarrow S \longleftarrow \begin{matrix} S(-1) \\ \oplus \\ S(-2) \end{matrix} \longleftarrow S(-3) \longleftarrow 0,$$

$$\beta(M) = \begin{pmatrix} 1 & 1 & - \\ - & 1 & 1 \end{pmatrix}.$$

Then

$$H_M(l) = H_S(l) - H_{\begin{smallmatrix} S(-1) \\ \oplus \\ S(-2) \end{smallmatrix}}(l) + H_{S(-3)}(l).$$

Breaking up the direct sums, we get $H_M(l) = H_S(l) - (H_{S(-1)}(l) + H_{S(-2)}(l)) + H_{S(-3)}(l)$, where $H_S(l) = \dim_k S_l$ is the dimension of the l^{th} graded piece of S, i.e. the number of degree l monomials in S. By the above lemma, we have $H_{S(i)}(l) = \dim_k(S_l(i)) = \binom{l+i+1}{1}$. It follows that

$$H_M(l) = \binom{l+1}{1} - \binom{l-1+1}{1} - \binom{l-2+1}{1} + \binom{l-3+1}{1}.$$

Observe that $H_M(0) = 1, H_M(1) = 1$, and $H_M(n) = 0$ for $n \geq 2$. $\Diamond$

For any given M with resolution $F_{\bullet}$ and Betti diagram $\beta(M)$, we have

$$\begin{aligned} H_M(l) &= \sum_i (-1)^i H_{F_i}(l) \\ &= \sum_{i,j} (-1)^i \beta(M)_{ij} H_{S(-j)}(l), \end{aligned}$$

where

$$H_{S(-d)}(l) = \begin{cases} 0 & ; l < d \\ \binom{(l-d)+n-1}{n-1} & ; l \geq d \end{cases}.$$

We can extend this to get information about whether a diagram is a Betti diagram for some module M. Since it doesn't make sense to have a module with $\dim_k M < 0$, we can conclude that if $H_D(l)$ is negative for some $l \in \mathbb{Z}$ then D is not a Betti diagram.

Lemma 1.5.7 ([9])**.** *Let D be a diagram. If $H_D(l) < 0$ for some $l \in \mathbb{Z}$ then there is no module M such that $\beta(M) = D$, i.e. D is not a Betti diagram.*

We cannot conclude from Lemma 1.5.7 that the diagram from Example 1.5.6 is a Betti diagram. But Lemma 1.5.7 tells us that the diagram from Example 1.5.6 might be a Betti diagram.

2

Short exact sequences and Betti decompositions

With the information provided in Chapter 1, let's take another look at Question 1.2.1. We state it here for convenience.

Question 1.2.1. *Let R be a ring. Consider a short exact sequence of R-modules:*

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0.$$

Given the Betti decompositions of A and C, what can we conclude about the Betti decomposition of B?

2.1 A class of Betti diagrams

Recall from Example 1.4.6 that

$$\beta = \begin{pmatrix} 1 & 1 & - \\ - & 1 & 1 \end{pmatrix} = 2 \begin{pmatrix} \frac{1}{3} & \frac{1}{2} & - \\ - & - & \frac{1}{6} \end{pmatrix} + 2 \begin{pmatrix} \frac{1}{6} & - & - \\ - & \frac{1}{2} & \frac{1}{3} \end{pmatrix}.$$

It turns out that this can be extended to all Betti diagrams of the form $\begin{pmatrix} n & n & - \\ - & n & n \end{pmatrix}$.

Theorem 2.1.1. *Any Betti diagram of the form* $\beta = \begin{pmatrix} n & n & - \\ - & n & n \end{pmatrix}$ *can be written as*

$$\beta = 2n \begin{pmatrix} \frac{1}{3} & \frac{1}{2} & - \\ - & - & \frac{1}{6} \end{pmatrix} + 2n \begin{pmatrix} \frac{1}{6} & - & - \\ - & \frac{1}{2} & \frac{1}{3} \end{pmatrix}.$$

Proof. We will prove this using Algorithm 1.4.5. Note that since we are looking at Betti diagrams of the same form, the initial degree sequence and pure diagram will always be $d_0 = (0, 1, 3)$ and $\pi(d_0) = \begin{pmatrix} \frac{1}{3} & \frac{1}{2} & - \\ - & - & \frac{1}{6} \end{pmatrix}$. The largest $a \in \mathbb{Z}^+$ for which each entry in $\beta - a\pi(d_0)$ is positive is the largest positive integer that satisfies the inequalities $n - \frac{a}{3} \geq 0, n - \frac{a}{2} \geq 0$, and $n - \frac{a}{6} \geq 0$. So $a = 2n$. Then $\beta - 2n\pi(d_0) = \begin{pmatrix} \frac{n}{3} & - & - \\ - & n & \frac{2n}{3} \end{pmatrix}$.

Now we have another degree sequence $d_1 = (0, 2, 3)$ and pure diagram $\pi(d_1) = \begin{pmatrix} \frac{1}{6} & - & - \\ - & \frac{1}{2} & \frac{1}{3} \end{pmatrix}$. (Note that any Betti diagram of the same form as β will have the same first degree sequence d_1 and pure diagram $\pi(d_1)$.) Observe that $2n\pi(d_1) = \beta - 2n\pi(d_0)$. Then $\beta - 2n\pi(d_0) - 2n\pi(d_1) = 0$. It follows that $\beta = 2n\pi(d_0) + 2n\pi(d_1)$. □

Example 2.1.2. Let $R = k[x, y]$. Consider the finitely generated graded R-module $M = \dfrac{k[x, y]}{(x, y^2)}$ from Example 1.2.12. Using Macaulay2 [8], we find that resolution of $M \oplus M$ is

$$0 \longleftarrow \begin{matrix} M \\ \oplus \\ M \end{matrix} \longleftarrow \begin{matrix} R \\ \oplus \\ R \end{matrix} \longleftarrow \begin{matrix} R(-1) \\ \oplus \\ R(-2) \\ \oplus \\ R(-1) \\ \oplus \\ R(-2) \end{matrix} \longleftarrow \begin{matrix} R(-3) \\ \oplus \\ R(-3) \end{matrix} \longleftarrow 0,$$

which has the Betti diagram $\beta(M \oplus M) = \begin{pmatrix} 2 & 2 & - \\ - & 2 & 2 \end{pmatrix}$. By Theorem 2.1.1, the Betti decomposition algorithm of $\beta(M \oplus M)$ gives us

$$\beta(M \oplus M) = 4 \begin{pmatrix} \frac{1}{3} & \frac{1}{2} & - \\ - & - & \frac{1}{6} \end{pmatrix} + 4 \begin{pmatrix} \frac{1}{6} & - & - \\ - & \frac{1}{2} & \frac{1}{3} \end{pmatrix}.$$

So, looking at the short exact sequence

$$0 \longrightarrow \frac{k[x,y]}{(x,y^2)} \longrightarrow \frac{k[x,y]}{(x,y^2)} \oplus \frac{k[x,y]}{(x,y^2)} \longrightarrow \frac{k[x,y]}{(x,y^2)} \longrightarrow 0,$$

we find that the Betti decomposition of the middle module is the sum of the Betti decompositions of the two outer modules. $\diamondsuit$

2.2 Direct sums of finitely generated graded R-modules in short exact sequences, and their Betti decompositions

Consider the short exact sequence of finitely generated graded R-modules M, N:

$$0 \longrightarrow M \longrightarrow M \oplus N \longrightarrow N \longrightarrow 0.$$

We want to describe the relationship between $\beta(M \oplus N)$ and $\beta(M), \beta(N)$. In Corollary 2.2.3, we find that

$$\beta(M) + \beta(N) = \beta(M \oplus N).$$

In the case where

$$\beta(M) = \beta(N) = \begin{pmatrix} n & n & - \\ - & n & n \end{pmatrix},$$

there is also a nice relationship between the Betti decompositions of $\beta(M), \beta(N)$ and the Betti decomposition of $\beta(M \oplus N)$.

Proposition 2.2.1. *Let R be a ring. Let f, g be R-module homomorphisms. Then*

$$ker(f) \oplus ker(g) = ker(f \oplus g)$$

and

$$im(f) \oplus im(g) = im(f \oplus g).$$

Proof. Let R be a ring. Let A, B, C, D be R-modules such that $A \cap C = \emptyset$ and $B \cap D = \emptyset$. Let $f\colon A \longrightarrow B$ and $g\colon C \longrightarrow D$ be R-module homomorphisms. Then $A \oplus C$ and $B \oplus D$ are

R-modules. So, we have the R-module homomorphism

$$f \oplus g \colon A \oplus C \longrightarrow B \oplus D$$
$$(a, c) \mapsto (f(a), g(c)).$$

We want to show that

$$\ker(f) \oplus \ker(g) = \ker(f \oplus g) \tag{2.2.1}$$

and

$$\mathrm{im}(f) \oplus \mathrm{im}(g) = \mathrm{im}(f \oplus g). \tag{2.2.2}$$

To prove (2.2.1), it suffices to show that

$$\ker(f) \oplus \ker(g) \subseteq \ker(f \oplus g)$$

and

$$\ker(f \oplus g) \subseteq \ker(f) \oplus \ker(g).$$

Let $(a, c) \in \ker(f) \oplus \ker(g)$. Then $a \in \ker(f)$ and $c \in \ker(g)$. Then we have $f(a) = 0$ and $g(c) = 0$. So by definition of $f \oplus g$, it follows that $(f \oplus g)(a, c) = (f(a), g(c)) = (0, 0)$. Therefore $(a, c) \in \ker(f \oplus g)$. It follows that $\ker(f) \oplus \ker(g) \subseteq \ker(f \oplus g)$.

Let $(a, c) \in \ker(f \oplus g)$. Using a similar argument in the reverse direction, we find that $(a, c) \in \ker(f) \oplus \ker(g)$. Therefore $\ker(f \oplus g) \subseteq \ker(f) \oplus \ker(g)$. Then (2.2.1) follows.

To prove (2.2.2), it suffices to show that

$$\mathrm{im}(f) \oplus \mathrm{im}(g) \subseteq \mathrm{im}(f \oplus g)$$

and

$$\mathrm{im}(f \oplus g) \subseteq \mathrm{im}(f) \oplus \mathrm{im}(g).$$

Let $(b, d) \in \mathrm{im}(f) \oplus \mathrm{im}(g)$. Then $b \in \mathrm{im}(f)$ and $d \in \mathrm{im}(g)$. It follows that $b = f(a), d = g(c)$ for some $a \in A, c \in C$. By our definition of $f \oplus g$, we have that $(f(a), g(c)) \in \mathrm{im}(f \oplus g)$. It follows that $(b, d) \in \mathrm{im}(f \oplus g)$. Thus $\mathrm{im}(f) \oplus \mathrm{im}(g) \subseteq \mathrm{im}(f \oplus g)$.

Let $(b, d) \in \operatorname{im}(f \oplus g)$. Following the same argument in the reverse direction, it follows that $(b, d) \in \operatorname{im}(f) \oplus \operatorname{im}(g)$. Then $\operatorname{im}(f \oplus g) \subseteq \operatorname{im}(f) \oplus \operatorname{im}(g)$, so (2.2.2) follows. $\square$

We use Proposition 2.2.1 to prove Proposition 2.2.2.

Proposition 2.2.2. *Let $R = k[x_1, \dots, x_n]$ be a ring over a field k. Let M, N be a finitely generated graded free R-modules. Let $F_\bullet$ be the minimal free resolution of M and let $G_\bullet$ be the minimal free resolution of N. Then the minimal free resolution of $M \oplus N$ is given by*

$$H_\bullet : \qquad 0 \longleftarrow M \oplus N \longleftarrow H_0 \longleftarrow H_1 \longleftarrow \cdots \longleftarrow H_l,$$

where $H_i := F_i \oplus G_i$.

Proof. Let $R = k[x_1, \dots, x_n]$ be a ring over a field k. Let M, N be a finitely generated graded free R-modules such that $M \cap N = \emptyset$. Let M be of length l_M and let m be a minimal generating set of M. Let $F_\bullet$ be the graded minimal free resolution of M constructed as in Remark 1.3.9,

$$\begin{tikzcd}
F_\bullet: & 0 & M \arrow[l] & F_0 \arrow[l, "\gamma_0"'] & & F_1 \arrow[ll, "\phi_0"'] \arrow[dl, "\gamma_1"] & & \cdots \arrow[ll, "\phi_1"'] \arrow[dl, "\gamma_2"] & & F_{l_M}. \arrow[ll, "\phi_{(l_{M-1})}"'] \arrow[dl, "\gamma_{(l_M)}"] \\
& & & & \ker(\gamma_0) \arrow[ul, "f_0"'] \arrow[dl] & & \ker(\gamma_1) \arrow[ul, "f_1"'] \arrow[dl] & & \ker(\gamma_{l_{M-1}}) \arrow[ul, "f_{l_{M-1}}"'] \arrow[dl] & \\
& & & 0 & & 0 \arrow[ul] & & \cdots \arrow[ul] & & 0 \arrow[ul]
\end{tikzcd}$$

Similarly, let N be a finitely generated graded free R-module of length l_N and let n be a minimal generating set. Let $G_\bullet$ be the graded minimal free resolution of N constructed as in Remark 1.3.9,

$$\begin{tikzcd}
G_\bullet: & 0 & N \arrow[l] & G_0 \arrow[l, "\psi_0"'] & & G_1 \arrow[ll, "\rho_0"'] \arrow[dl, "\psi_1"] & & \cdots \arrow[ll, "\rho_1"'] \arrow[dl, "\psi_2"] & & G_{l_N}. \arrow[ll, "\rho_{(l_{N-1})}"'] \arrow[dl, "\psi_{(l_N)}"] \\
& & & & \ker(\psi_0) \arrow[ul, "g_0"'] \arrow[dl] & & \ker(\psi_1) \arrow[ul, "g_1"'] \arrow[dl] & & \ker(\psi_{l_{N-1}}) \arrow[ul, "g_{l_{N-1}}"'] \arrow[dl] & \\
& & & 0 & & 0 \arrow[ul] & & \cdots \arrow[ul] & & 0 \arrow[ul]
\end{tikzcd}$$

It follows that F_0, G_0 are finitely generated free R-modules constructed using the degrees of the minimal generating sets m, n of M, N, respectively. And for $i \geq 1$, each F_i, G_i is a finitely generated free R-module constructed using the degrees of the minimal generating sets p_i, q_i of $\ker(\gamma_i), \ker(\phi_i)$, respectively. Refer to Theorem 1.3.5 and Remark 1.3.9 for more detail regarding the construction of F_i, G_i using the degrees of the sets p_i, q_i.

In order to prove Proposition 2.2.2, we will construct the graded minimal free resolution $H_\bullet$ of $M \oplus N$. As we describe this construction, we will show that each $H_i = F_i \oplus G_i$ in the graded minimal free resolution of $M \oplus N$.

Consider $M \oplus N$. Since m, n are minimal generating sets of M, N, respectively, it follows that $m \oplus n$ is a minimal generating set of $M \oplus N$. By Theorem 1.3.5, there is a finitely generated free module H_0 constructed using the degrees of elements of $m \oplus n$ and the surjective graded R-module homomorphism $\pi_0 \colon H_0 \longrightarrow M \oplus N$. Recall that we have the surjective graded R-module homomorphisms $\gamma_0 \colon F_0 \longrightarrow M$ and $\psi_0 \colon G_0 \longrightarrow N$ where F_0, G_0 were constructed using the degrees of m, n, respectively. It follows that $H_0 = F_0 \oplus G_0$ and that

$$\pi_0 = \gamma_0 \oplus \psi_0 \colon F_0 \oplus G_0 \longrightarrow M \oplus N.$$

Then we have the short exact sequence

$$0 \longrightarrow \ker(\pi_0) \xrightarrow{\delta_0} H_0 \xrightarrow{\pi_0} M \oplus N \longrightarrow 0.$$

By Proposition 2.2.1, we have that

$$\begin{aligned} \ker(\pi_0) &= \ker(\gamma_0 \oplus \psi_0) \\ &= \ker(\gamma_0) \oplus \ker(\psi_0). \end{aligned}$$

Recall that p_0, q_0 are minimal generating sets of $\ker(\gamma_0), \ker(\psi_0)$, respectively. It follows that $p_0 \oplus q_0$ is a minimal generating set of $\ker(\pi_0)$. By Theorem 1.3.5, there is a finitely generated free R-module H_1 and a graded surjective R-module homomorphism

$$\pi_1 \colon H_1 \longrightarrow \ker(\pi_0)$$

such that H_1 was constructed using the degrees of $p_0 \oplus q_0$. Recall that we have the graded surjective R-module homomorphisms $\gamma_1 \colon F_1 \longrightarrow \ker(\gamma_0)$ and $\psi_1 \colon G_1 \longrightarrow \ker(\psi_0)$ where p_0, q_0 are minimal generating sets of $\ker(\gamma_0), \ker(\psi_0)$, respectively, and F_1, G_1 are constructed using the degrees of p_0, q_0, respectively. It follows that $H_1 = F_1 \oplus G_1$ and that

$$\pi_1 = \gamma_1 \oplus \psi_1 \colon F_1 \oplus G_1 \longrightarrow \ker(\gamma_0) \oplus \ker(\psi_0) = \ker(\pi_0).$$

Then we can define the R-module homomorphism

$$\sigma_0 := \delta_0 \circ \pi_1 \colon H_1 \longrightarrow H_0.$$

Then we have

$$\begin{tikzcd}
0 & M \oplus N \arrow[l] & H_0 \arrow[l, "\pi_0"'] & & H_1 \arrow[ll, "\sigma_1"'] \arrow[dl, "\pi_1"] . \\
& & & \ker(\pi_0) \arrow[ul, "\delta_0"'] \arrow[dl] & \\
& & 0 & & 0 \arrow[ul]
\end{tikzcd}$$

Observe that

$$\begin{tikzcd}
0 \arrow[r] & \ker(\pi_1) \arrow[r, "\delta_0"] & H_1 \arrow[r, "\pi_1"] & \ker(\pi_0) \arrow[r] & 0
\end{tikzcd}$$

is a short exact sequence. Recall that $\pi_1 = \gamma_1 \oplus \psi_1$. Therefore

$$\begin{aligned} \ker(\pi_1) &= \ker(\gamma_1 \oplus \psi_1) \\ &= \ker(\gamma_1) \oplus \ker(\psi_1). \end{aligned}$$

Recall that $\ker(\gamma_1), \ker(\psi_1)$ are finitely generated by minimal generating sets p_1, q_1, respectively. It follows that $p_1 \oplus q_1$ is a minimal generating set of $\ker(\gamma_1) \oplus \ker(\psi_1)$. Therefore $p_1 \oplus q_1$ is a minimal generating set of $\ker(\pi_1)$. Then we have a surjective R-module homomorphism $\pi_2 \colon H_2 \longrightarrow \ker(\pi_1)$ where H_2 is constructed using the degrees of $p_1 \oplus q_1$.

Recall that F_2, G_2 are constructed using the degrees of p_1, q_1, respectively. It follows that $H_2 = F_2 \oplus G_2$.

We can continue in this manner to find the surjective R-module homomorphisms

$$\pi_i \colon H_i = F_i \oplus G_i \longrightarrow \ker(\pi_{i-1}) = \ker(\gamma_{i-1}) \oplus \ker(\psi_{i-1})$$

for $i \leq \max l_M, l_N$. $\square$

The following corollary is a direct result of Proposition 2.2.2.

Corollary 2.2.3. *Let $R = k[x_1, \dots, x_n]$ be a ring over a field k. Let M, N be a finitely generated graded free R-modules. Given the short exact sequence*

$$0 \longrightarrow M \longrightarrow M \oplus N \longrightarrow N \longrightarrow 0,$$

then $\beta(M) + \beta(N) = \beta(M \oplus N)$.

For the specific class of Betti diagrams described in Section 2.1, we have a further result that relates not only the Betti diagrams of the modules in a short exact sequence, but also their Betti decompositions.

Proposition 2.2.4. *Let $R = k[x, y]$ be a ring over a field k. Let M be a finitely generated free graded R-module with the Betti diagram $\beta(M) = \begin{pmatrix} n & n & - \\ - & n & n \end{pmatrix}$. Given the short exact sequence*

$$0 \longrightarrow M \longrightarrow M \oplus M \longrightarrow M \longrightarrow 0,$$

the sum of the Betti decompositions of the two outer modules is the Betti decomposition of $M \oplus M$.

Proof. Let $R = k[x, y]$ be a ring over a field k. Let M be a finitely generated free graded R-module with the Betti diagram $\beta(M) = \begin{pmatrix} n & n & - \\ - & n & n \end{pmatrix}$.

By Corollary 2.2.3, we have that

$$\beta(M)+\beta(M)=\beta(M\oplus M)=\begin{pmatrix}2n & 2n & - \\ - & 2n & 2n\end{pmatrix}$$

. By Theorem 2.1.1, we have the Betti decompositions of M and $M\oplus M$:

$$\begin{aligned}\beta(M) &= 2n\begin{pmatrix}\frac{1}{3} & \frac{1}{2} & - \\ - & - & \frac{1}{6}\end{pmatrix} + 2n\begin{pmatrix}\frac{1}{6} & - & - \\ - & \frac{1}{2} & \frac{1}{3}\end{pmatrix}, \\ \beta(M\oplus M) &= 4n\begin{pmatrix}\frac{1}{3} & \frac{1}{2} & - \\ - & - & \frac{1}{6}\end{pmatrix} + 4n\begin{pmatrix}\frac{1}{6} & - & - \\ - & \frac{1}{2} & \frac{1}{3}\end{pmatrix}.\end{aligned}$$

Observe that

$$\begin{aligned} & 2n\begin{pmatrix}\frac{1}{3} & \frac{1}{2} & - \\ - & - & \frac{1}{6}\end{pmatrix} + 2n\begin{pmatrix}\frac{1}{6} & - & - \\ - & \frac{1}{2} & \frac{1}{3}\end{pmatrix} \\ + \quad & 2n\begin{pmatrix}\frac{1}{3} & \frac{1}{2} & - \\ - & - & \frac{1}{6}\end{pmatrix} + 2n\begin{pmatrix}\frac{1}{6} & - & - \\ - & \frac{1}{2} & \frac{1}{3}\end{pmatrix} \\ = \quad & 4n\begin{pmatrix}\frac{1}{3} & \frac{1}{2} & - \\ - & - & \frac{1}{6}\end{pmatrix} + 4n\begin{pmatrix}\frac{1}{6} & - & - \\ - & \frac{1}{2} & \frac{1}{3}\end{pmatrix}.\end{aligned}$$

Thus the Betti decomposition of M added to the Betti decomposition of M is the Betti decomposition of $M\oplus M$. $\square$

3

Complete intersections

Recall from Section 1.2 the second main question, stated here for convenience:

Question 1.2.2. *Let $S = k[x_1, \ldots, x_d]$ be a polynomial ring over a field k and let $I = (f_1, \ldots, f_d)$ be an ideal of S generated by a homogeneous regular sequence with deg$(f_i) = e_i$. What is the Betti decomposition of S/I in terms of the degrees e_i?*

In [6], this question was posed an answered up to codimension ≤ 3 (see Proposition 3.1.4).

3.1 Complete intersections in codimension ≤ 3

Before going into the exciting new answers to Question 1.2.2, we must first define the new terms from Question 1.2.2.

Definition 3.1.1. Let R be a standard graded ring. Then $f_1, \ldots, f_d$ is a homogeneous regular sequence on R if

i. f_i is homogeneous for all $1 \leq i \leq d$,

ii. the ideal $(f_1, \ldots, f_d) \neq R$,

iii. there is no non-zero $g \in R$ such that $f_1 \cdot g = 0$, and

iv. for all $2 \leq i \leq d$, there is no non-zero $g \in R/(f_1, \dots, f_{i-1})$ such that $g \cdot f_i = 0$.

$\triangle$

Example 3.1.2. Let $R = k[x, y, z]$. Then x^4, y^7, z^8 is a homogeneous regular sequence on R. Observe that x^4, y^7, z^8 are all homogeneous. Consider the ideal $(x^4, y^7, z^8) \subseteq R$. Note that $xy \in R$ and $xy \notin (x^4, y^7, z^8)$, so $(x^4, y^7, z^8) \neq R$. Observe that $x^4 \cdot r \neq 0$ for all non-zero $r \in R$. Consider $\overline{s} \in R/(x^4)$, $\overline{s} \neq \overline{0}$. Then $\overline{s} = s + (x^4)$, and $y^7 \cdot \overline{s} = y^7 \cdot s + (x^4)$. Since $\overline{s} \neq \overline{0}$, it follows that s is not a multiple of x^4. This forces $y^7 \cdot \overline{s} \neq \overline{0}$. Consider $\overline{t} \in R/(x^4, y^7)$, $\overline{t} \neq \overline{0}$. Then $\overline{t} = t + (x^4, y^7)$, and $z^8 \cdot t \notin (x^4, y^7)$. So $z^8 \cdot \overline{t} \neq 0$. $\diamondsuit$

Definition 3.1.3. Let $S = k[x_1, \dots, x_n]$ be a polynomial ring over a field k. Let $f_1, \dots, f_d$ be a homogeneous regular sequence. If $I = (f_1, \dots, f_d)$, then the ring S/I is called a *graded complete intersection.* $\triangle$

We say that a complete intersection of the form $k[x_1, \dots, x_d]/(f_1, \dots, f_d)$ is *in codimension* d.

The initial answer to Question 1.2.2 from [6] is restated in Proposition 3.1.4.

Proposition 3.1.4 ([6])**.** *Let $S = k[x_1, \dots, x_d]$ be a polynomial ring over a field k and let $I = (f_1, \dots, f_d)$ be an ideal of S generated by a homogeneous regular sequence with deg$(f_i) = e_i$. If $d \leq 3$, then the Betti decomposition of S/I obtained from Algorithm 1.4.5 is completely determined by the degrees $e_1, \dots, e_d$. In particular, for*

$d = 1: \quad \beta(S/I) = e_1 \cdot \pi(0, e_1)$

$d = 2: \quad \beta(S/I) = e_1e_2 \cdot \pi(0, e_1, e_1 + e_2) + e_1e_2 \cdot \pi(0, e_2, e_1 + e_2)$

$d = 3:$ *If* $e_1 \leq e_2 \leq e_3$, *then*

$$\begin{aligned}\beta(S/I) = \quad & e_1e_2(e_2 + e_3) \cdot \pi(0, e_1, e_1 + e_2, e_1 + e_2 + e_3)\\ & + e_1e_2(e_3 - e_1) \cdot \pi(0, e_2, e_1 + e_2, e_1 + e_2 + e_3)\\ & + 2e_1e_2(e_1 + e_3 - e_2) \cdot \pi(0, e_2, e_1 + e_3, e_1 + e_2 + e_3)\\ & + e_1e_2(e_3 - e_1) \cdot \pi(0, e_3, e_1 + e_3, e_1 + e_2 + e_3)\\ & + e_1e_2(e_2 + e_3) \cdot \pi(0, e_3, e_2 + e_3, e_1 + e_2 + e_3)\end{aligned}$$

In Section 3.2, we extend Proposition 3.1.4 to $d = 4$. This will allow us to describe the Betti decomposition of complete intersections of the form $S = k[x, y, z, w]/(x^{e_1}, y^{e_2}, z^{e_3}, w^{e_4})$, for $e_1, e_2, e_3, e_4 \in \mathbb{Z}^+$. We consider the the first five of the following cases:

(i) $e_1 = e_2 = e_3 = e_4$,

(ii) $e_1 = e_2 = e_3 < e_4$,

(iii) $e_1 = e_2 < e_3 = e_4$,

(iv) $e_1 < e_2 = e_3 = e_4$,

(v) $e_1 = e_2 < e_3 < e_4$,

(vi) $e_1 < e_2 = e_3 < e_4$,

(vii) $e_1 < e_2 < e_3 = e_4$,

(viii) $e_1 < e_2 < e_3 < e_4$.

3.2 Cases of complete intersections in codimension 4

We will consider cases $(i), (ii), (iii), (iv)$, and (v), as stated at the end of the previous section. In the following proofs of Propositions 3.2.1, 3.2.2, 3.2.3, and 3.2.4, we omit the entries in the first and last columns in the Betti decomposition algorithm because these entries will always be eliminated in the final step of the algorithm.

Proposition 3.2.1. *Let $S = k[x, y, z, w]$ be a polynomial ring over a field k and let $I = (x^\alpha, y^\alpha, z^\alpha, w^\alpha)$ be an ideal of S generated by a homogeneous regular sequence. Then the Betti decomposition of S/I obtained from Algorithm 1.4.5 is given by*

$$\beta(S/I) = 24\alpha^4\pi(0, \alpha, 2\alpha, 3\alpha, 4\alpha).$$

Proof. Let $S = k[x, y, z, w]$ be a polynomial ring over a field k and let $I = (x^\alpha, y^\alpha, z^\alpha, w^\alpha)$ be an ideal of S generated by a homogeneous regular sequence. We construct the graded minimal free resolution of S/I as described in Remark 1.3.9:

$$S/I \longleftarrow S \longleftarrow S^4(-\alpha) \longleftarrow S^6(-2\alpha) \longleftarrow S^4(-3\alpha) \longleftarrow S(-4\alpha) \longleftarrow 0.$$

The corresponding Betti diagram $\beta(S/I)$, denoted β, has nonzero entries given by

$$\beta_{0,0} = 1, \beta_{1,\alpha} = 4, \beta_{2,2\alpha} = 6, \beta_{3,3\alpha} = 4, \beta_{4,4\alpha} = 1.$$

We use Algorithm 1.4.5 to find the Betti decomposition of S/I. The first and only degree sequence for the decomposition of this Betti diagram is $d = (0, \alpha, 2\alpha, 3\alpha, 4\alpha)$ with corresponding elimination matrix $\pi(d)$ with nonzero entries:

$$\pi(d)_{1,\alpha} = \frac{1}{6\alpha^4}$$
$$\pi(d)_{2,2\alpha} = \frac{1}{4\alpha^4}$$
$$\pi(d)_{3,3\alpha} = \frac{1}{6\alpha^4}.$$

Observe that $\beta = 24\alpha^4\pi(d)$. □

Proposition 3.2.2. *Let $S = k[x, y, z, w]$ be a polynomial ring over a field k and let $I = (x^\alpha, y^\alpha, z^\alpha, w^\delta)$ be an ideal of S generated by a homogeneous regular sequence. Let $\alpha < \delta$. Then the Betti decomposition of S/I obtained from Algorithm 1.4.5 is given by*

$$\begin{aligned}\beta(S/I) &= 6\alpha^3\delta\pi(0, \alpha, 2\alpha, 3\alpha, 3\alpha + \delta)\\ &\quad +6\alpha^3\delta\pi(0, \alpha, 2\alpha, 2\alpha + \delta, 3\alpha + \delta)\\ &\quad +6\alpha^3\delta\pi(0, \alpha, \alpha + \delta, 2\alpha + \delta, 3\alpha + \delta)\\ &\quad +6\alpha^3\delta\pi(0, \delta, \alpha + \delta, 2\alpha + \delta, 3\alpha + \delta).\end{aligned}$$

Proof. Let $S = k[x, y, z, w]$ be a polynomial ring over a field k and let $I = (x^\alpha, y^\alpha, z^\alpha, w^\delta)$ be an ideal of S generated by a homogeneous regular sequence. Let $\alpha < \delta$. We construct the graded minimal free resolution of S/I as in Remark 1.3.9:

$$S/I \longleftarrow S \longleftarrow \begin{matrix} S^3(-\alpha)\\ \oplus\\ S(-\delta)\end{matrix} \longleftarrow \begin{matrix} S^3(-2\alpha)\\ \oplus\\ S^3(-\alpha-\delta)\end{matrix} \longleftarrow \begin{matrix} S(-3\alpha)\\ \oplus\\ S^3(-2\alpha-\delta)\end{matrix} \longleftarrow S(-3\alpha-\delta) \longleftarrow 0.$$

Then the Betti diagram of S/I, denoted $\beta^{(0)}$, has nonzero entries

$$\beta^{(0)}_{0,0} = 1, \beta^{(0)}_{1,\alpha} = 3, \beta^{(0)}_{1,\delta} = 1, \beta^{(0)}_{2,2\alpha} = 3, \beta^{(0)}_{2,\alpha+\delta} = 3, \beta^{(0)}_{3,3\alpha} = 1, \beta^{(0)}_{3,2\alpha+\delta} = 3, \beta^{(0)}_{4,3\alpha+\delta} = 1.$$

We follow Algorithm 1.4.5 to find the Betti decomposition of $\beta^{(0)}$. The first degree sequence in the decomposition of $\beta^{(0)}$ is given by $d_0 = (0, \alpha, 2\alpha, 3\alpha, 3\alpha+\delta)$. Then the nonzero entries of the first elimination matrix are given by $\pi(d_0)$:

$$\begin{aligned}\pi(d_0)_{1,\alpha} &= \frac{1}{2\alpha^3(2\alpha+\delta)}\\ \pi(d_0)_{2,2\alpha} &= \frac{1}{2\alpha^3(\alpha+\delta)}\\ \pi(d_0)_{3,3\alpha} &= \frac{1}{6\alpha^3\delta}.\end{aligned}$$

We want to construct a new $\beta^{(1)} = \beta^{(0)} - x_0\pi(d_0)$ by substracting a scalar multiple $x_0 \in \mathbb{Z}^+$

of $\pi(d_0)$ from $\beta^{(0)}$ so that the result has entries 0 or greater than 0 and that we eliminate one of the nonzero entries from $\beta^{(0)}$. This scalar will be the largest x_0 that satisfies the following inequalities:

$$\begin{aligned}
3 - \frac{x_0}{2\alpha^3(2\alpha+\delta)} &\geq 0 \\
3 - \frac{x_0}{2\alpha^3(\alpha+\delta)} &\geq 0 \\
1 - \frac{x_0}{6\alpha^3\delta} &\geq 0.
\end{aligned}$$

We see that $6\alpha^3\delta$ satisfies the above inequalities. Then $\beta^{(1)} = \beta^{(0)} - 6\alpha^3\delta\pi(d_0)$ has nonzero entries given by

$$\begin{aligned}
\beta^{(1)}_{1,\alpha} &= \frac{6\alpha}{2\alpha+\delta} \\
\beta^{(1)}_{1,\delta} &= 1 \\
\beta^{(1)}_{2,2\alpha} &= \frac{3\alpha}{\alpha+\delta} \\
\beta^{(1)}_{2,\alpha+\delta} &= 3 \\
\beta^{(1)}_{3,3\alpha} &= 0 \\
\beta^{(1)}_{3,2\alpha+\delta} &= 3.
\end{aligned}$$

Following Algorithm 1.4.5, we have the degree sequence $d_1 = (0, \alpha, 2\alpha, 2\alpha+\delta, 3\alpha+\delta)$ and nonzero entries of the corresponding elimination matrix $\pi(d_1)$:

$$\begin{aligned}
\pi(d_1)_{1,\alpha} &= \frac{1}{\alpha^2(\alpha+\delta)(2\alpha+\delta)} \\
\pi(d_1)_{2,2\alpha} &= \frac{1}{2\alpha^2\delta(\alpha+\delta)} \\
\pi(d_1)_{3,2\alpha+\delta} &= \frac{1}{\alpha\delta(\alpha+\delta)(2\alpha+\delta)}.
\end{aligned}$$

We find that the largest x_1 such that each entry of $\beta^{(2)} = \beta^{(1)} - x_1\pi(d_1)$ is greater than or equal to 0 is $6\alpha^3\delta$:

$$
\begin{aligned}
\frac{6\alpha}{2\alpha+\delta} - \frac{x_1}{\alpha^2(\alpha+\delta)(2\alpha+\delta)} &\geq 0 \\
\frac{3\alpha}{\alpha+\delta} - \frac{x_1}{2\alpha^2\delta(\alpha+\delta)} &\geq 0 \\
3 - \frac{x_1}{\alpha\delta(\alpha+\delta)(2\alpha+\delta)} &\geq 0.
\end{aligned}
$$

Then the nonzero entries of $\beta^{(2)}$ are given by

$$
\begin{aligned}
\beta^{(2)}_{0,0} &= \frac{6\alpha^2}{(2\alpha+\delta)(3\alpha+\delta)} \\
\beta^{(2)}_{1,\alpha} &= \frac{6\alpha^2}{(\alpha+\delta)(2\alpha+\delta)} \\
\beta^{(2)}_{1,\delta} &= 1 \\
\beta^{(2)}_{2,2\alpha} &= 0 \\
\beta^{(2)}_{2,\alpha+\delta} &= 3 \\
\beta^{(2)}_{3,2\alpha+\delta} &= \frac{3\delta(3\alpha+\delta)}{(\alpha+\delta)(2\alpha+\delta)} \\
\beta^{(2)}_{4,3\alpha+\delta} &= \frac{\delta(5\alpha^2+6\alpha\delta+\delta^2)}{(\alpha+\delta)(2\alpha+\delta)(3\alpha+\delta)}.
\end{aligned}
$$

Then the next degree sequence is $d_2 = (0, \alpha, \alpha+\delta, 2\alpha+\delta, 3\alpha+\delta)$ and the nonzero entries of the elimination matrix $\pi(d_2)$ are

$$
\begin{aligned}
\pi(d_2)_{1,\alpha} &= \frac{1}{\alpha\delta(\alpha+\delta)(2\alpha+\delta)} \\
\pi(d_2)_{2,\alpha+\delta} &= \frac{1}{2\alpha^2\delta(\alpha+\delta)} \\
\pi(d_2)_{3,2\alpha+\delta} &= \frac{1}{\alpha^2(\alpha+\delta)(2\alpha+\delta)}.
\end{aligned}
$$

We find that $6\alpha^3$ is the largest x_2 such that $\beta^{(3)} = \beta^{(2)} - x_2\delta\pi(d_2)$ has entries that are greater than or equal to zero. The nonzero entries of $\beta^{(3)}$ are:

$$\begin{aligned}
\beta^{(3)}_{1,\alpha} &= 0 \\
\beta^{(3)}_{1,\delta} &= 1 \\
\beta^{(3)}_{2,\alpha+\delta} &= \frac{3\delta}{\alpha+\delta} \\
\beta^{(3)}_{3,2\alpha+\delta} &= \frac{3\delta}{2\alpha+\delta}
\end{aligned}$$

Then we have our final degree sequence $d_3 = (0, \delta, \alpha+\delta, 2\alpha+\delta, 3\alpha+\delta)$ and the nonzero entries of the corresponding elimination matrix $\pi(d_3)$:

$$\begin{aligned}
\pi(d_3)_{1,\delta} &= \frac{1}{6\alpha^3\delta} \\
\pi(d_3)_{2,\alpha+\delta} &= \frac{1}{2\alpha^3(\alpha+\delta)} \\
\pi(d_3)_{3,2\alpha+\delta} &= \frac{1}{2\alpha^3(2\alpha+\delta)}.
\end{aligned}$$

Observe that $\beta^{(3)} = 6\alpha^3\delta\pi(d_3)$. Then

$$\begin{aligned}
\beta^{(0)} \quad = \quad & 6\alpha^3\delta\pi(0, \alpha, 2\alpha, 3\alpha, 3\alpha+\delta) \\
& +6\alpha^3\delta\pi(0, \alpha, 2\alpha, 2\alpha+\delta, 3\alpha+\delta) \\
& +6\alpha^3\delta\pi(0, \alpha, \alpha+\delta, 2\alpha+\delta, 3\alpha+\delta) \\
& +6\alpha^3\delta\pi(0, \delta, \alpha+\delta, 2\alpha+\delta, 3\alpha+\delta). \quad \square
\end{aligned}$$

Proposition 3.2.3. *Let $S = k[x, y, z, w]$ be a polynomial ring over a field k and let $I = (x^\alpha, y^\alpha, z^\delta, w^\delta)$ be an ideal of S generated by a homogeneous regular sequence. Let $\alpha < \delta$. Then the Betti decomposition of S/I obtained from Algorithm 1.4.5 is given by*

$$\begin{aligned}
\beta(S/I) \quad = \quad & 4\alpha^2\delta^2\pi(0, \alpha, 2\alpha, 2\alpha+\delta, 2\alpha+2\delta) \\
& +2\alpha^2\delta(\alpha+3\delta)\pi(0, \alpha, \alpha+\delta, 2\alpha+\delta, 2\alpha+2\delta) \\
& +4\alpha^2\delta(\delta-\alpha)\pi(0, \delta, \alpha+\delta, 2\alpha+\delta, 2\alpha+2\delta) \\
& +2\alpha^2\delta(\alpha+3\delta)\pi(0, \delta, \alpha+\delta, \alpha+2\delta, 2\alpha+2\delta) \\
& +4\alpha^2\delta^2\pi(0, \delta, 2\delta, \alpha+2\delta, 2\alpha+2\delta)
\end{aligned}$$

Proof. Let $S = k[x, y, z, w]$ be a polynomial ring over a field k and let $I = (x^\alpha, y^\alpha, z^\delta, w^\delta)$ be an ideal of S generated by a homogeneous regular sequence. Let $\alpha < \delta$. We construct the graded minimal free resolution of S/I as in Remark 1.3.9:

$$S/I \longleftarrow S \longleftarrow \begin{array}{c} S^2(-\alpha) \\ \oplus \\ S^2(-\delta) \end{array} \longleftarrow \begin{array}{c} S(-2\alpha) \\ \oplus \\ S^4(-\alpha-\delta) \\ \oplus \\ S(-2\delta) \end{array} \longleftarrow \begin{array}{c} S^2(-2\alpha-\delta) \\ \oplus \\ S^2(-\alpha-2\delta) \end{array} \longleftarrow S(-2\alpha-2\delta) \longleftarrow 0.$$

The nonzero entries of $\beta(S/I)$, denoted $\beta^{(0)}$, are

$$\beta^{(0)}_{0,0} = 1, \beta^{(0)}_{1,\alpha} = 2, \beta^{(0)}_{1,\delta} = 2,$$
$$\beta^{(0)}_{2,2\alpha} = 1, \beta^{(0)}_{2,\alpha+\delta} = 4\beta^{(0)}_{2,2\delta} = 1,$$
$$\beta^{(0)}_{3,2\alpha+\delta} = 2, \beta^{(0)}_{3,\alpha+2\delta} = 2, \beta^{(0)}_{4,2\alpha+2\delta} = 1.$$

The first degree sequence is $d_0 = (0, \alpha, 2\alpha, 2\alpha+\delta, 2\alpha+2\delta)$. The non-zero entries of the corresponding elimination matrix $\pi(d_0)$ are as follows:

$$\begin{aligned} \pi(d_0)_{1,\alpha} &= \frac{1}{\alpha^2(\alpha+\delta)(\alpha+2\delta)} \\ \pi(d_0)_{2,2\alpha} &= \frac{1}{4\alpha^2\delta^2} \\ \pi(d_0)_{3,2\alpha+\delta} &= \frac{1}{\delta^2(2\alpha+\delta)(\alpha+\delta)}. \end{aligned}$$

We want to find the largest x_0 such that $\beta^{(0)} - x_0\pi(d_0)$ has entries ≥ 0. So, we need to find the largest x_0 such that

$$x_0 \leq 2\alpha^4 + 6\alpha^3\delta + 4\alpha^2\delta^2$$
$$x_0 \leq 4\alpha^2\delta^2$$
$$x_0 4\alpha^2\delta^2 + 6\alpha\delta^3 + 2\delta^4.$$

We have that $4\alpha^2\delta^2 < 2\alpha^4 + 6\alpha^3\delta + 4\alpha^2\delta^2$ and $4\alpha^2\delta^2 < 4\alpha^2\delta^2 + 6\alpha\delta^3 + 2\delta^4$, so $x_0 = 4\alpha^2\delta^2$ is the largest solution. So, we choose $x_0 = 4\alpha^2\delta^2$ as the coefficient for $\pi(d_0)$. Let $\beta^{(1)} =$

$\beta^{(0)} - 4\alpha^2\delta^2\pi(d_0)$. Then $\beta^{(1)}$ has entries

$$\begin{aligned}
\beta^{(1)}_{1,\alpha} &= 2 - \frac{4\alpha^2\delta^2}{\alpha^2(\alpha+\delta)(\alpha+2\delta)} = \frac{2\alpha(\alpha+3\delta)}{(\alpha+\delta)(\alpha+2\delta)},\\
\beta^{(1)}_{2,2\alpha} &= 1 - \frac{4\alpha^2\delta^2}{4\alpha^2\delta^2} = 0,\\
\beta^{(1)}_{3,2\alpha+\delta} &= 2 - \frac{4\alpha^2\delta^2}{\delta^2(2\alpha+\delta)(\alpha+\delta)} = \frac{2\delta(3\alpha+\delta)}{(2\alpha+\delta)(\alpha+\delta)}.
\end{aligned}$$

Our next degree sequence is $d_1 = (0, \alpha, \alpha+\delta, 2\alpha+\delta, 2\alpha+2\delta)$ and has a corresponding elimination matrix $\pi(d_1)$ with non-zero entries

$$\begin{aligned}
\pi(d_1)_{1,\alpha} &= \frac{1}{\alpha\delta(\alpha+\delta)(\alpha+2\delta)},\\
\pi(d_1)_{2,\alpha+\delta} &= \frac{1}{\alpha\delta(\alpha+\delta)^2},\\
\pi(d_1)_{3,2\alpha+\delta} &= \frac{1}{\alpha\delta(\alpha+\delta)(2\alpha+\delta)}.
\end{aligned}$$

So, we need to find the largest x_1 such that

$$x_1 \le 2\alpha^2\delta(\alpha+3\delta) = 2\alpha^3\delta + 6\alpha^2\delta^2,$$

$$x_1 \le 4\alpha\delta(\alpha+\delta)^2 = 4\alpha^3\delta + 8\alpha^2\delta^2 + 4\alpha\delta^3,$$

$$x_1 \le 2\alpha\delta^2(3\alpha+\delta) = 6\alpha^2\delta^2 + 2\alpha\delta^3.$$

We see that $2\alpha^3\delta + 6\alpha^2\delta^2 < 4\alpha^3\delta + 8\alpha^2\delta^2 + 4\alpha\delta^3$ and $2\alpha^3\delta + 6\alpha^2\delta^2 < 6\alpha^2\delta^2 + 2\alpha\delta^3$, since $\alpha < \delta$. So, we choose $x_1 = 2\alpha^2\delta(\alpha+3\delta) = 2\alpha^3\delta + 6\alpha^2\delta^2$ as the coefficient for $\pi(d_1)$. Now let $\beta^{(2)} = \beta^{(1)} - 2\alpha^2\delta(\alpha+3\delta)\pi(d_1)$. Then $\beta^{(2)}$ has entries

$$\begin{aligned}
\beta^{(2)}_{1,\alpha} &= 0\\
\beta^{(2)}_{2,\alpha+\delta} &= 4 - \frac{2\alpha^2\delta(\alpha+3\delta)}{\alpha\delta(\alpha+\delta)^2} = \frac{2(\alpha^2+\alpha\delta+2\delta^2)}{(\alpha+\delta)^2}\\
\beta^{(2)}_{3,2\alpha+\delta} &= \frac{2\delta(3\alpha+\delta)}{(2\alpha+\delta)(\alpha+\delta)} - \frac{2\alpha^2\delta(\alpha+3\delta)}{\alpha\delta(\alpha+\delta)(2\alpha+\delta)} = \frac{2(\delta-\alpha)}{(2\alpha+\delta)}.
\end{aligned}$$

The next degree sequence is $d_2 = (0, \delta, \alpha + \delta, 2\alpha + \delta, 2\alpha + 2\delta)$, which has a corresponding elimination matrix $\pi(d_2)$ with the following entries:

$$\begin{aligned}
\pi(d_2)_{1,\delta} &= \frac{1}{2\alpha^2\delta(2\alpha+\delta)}, \\
\pi(d_2)_{2,\alpha+\delta} &= \frac{1}{\alpha^2(\alpha+\delta)^2}, \\
\pi(d_2)_{3,2\alpha+\delta} &= \frac{1}{2\alpha^2\delta(2\alpha+\delta)}.
\end{aligned}$$

We need to find the largest x_2 such that the entries of $\beta^{(2)} - x_2\pi(d_2)$ are greater than or equal to 0. So, we need the largest x_2 such that

$$\begin{aligned}
& x_2 \leq 4\alpha^2\delta(2\alpha+\delta) = 8\alpha^3\delta + 4\alpha^2\delta^2, \\
& x_2 \leq \alpha^2(2\alpha^2 + 2\alpha\delta + 4\delta^2) = 2\alpha^4 + 2\alpha^3\delta + 4\alpha^2\delta^2, \\
\text{and} \qquad & x_2 \leq 4\alpha^2\delta(\delta-\alpha) = 4\alpha^2\delta^2 - 4\alpha^3\delta.
\end{aligned}$$

Since $\alpha, \delta \geq 0$, it follows that $4\alpha^2\delta^2 - 4\alpha^3\delta < 8\alpha^3\delta + 4\alpha^2\delta^2$ and $4\alpha^2\delta^2 - 4\alpha^3\delta < 2\alpha^4 + 2\alpha^3\delta + 4\alpha^2\delta^2$. So, we choose $x_2 = 4\alpha^2\delta(\delta - \alpha) = 4\alpha^2\delta^2 - 4\alpha^3\delta$ as the coefficient for $\pi(d_2)$. Now, we let $\beta^{(3)} = \beta^{(2)} - 4\alpha^2\delta(\delta-\alpha)\pi(d_2)$, which has the following entries:

$$\begin{aligned}
\beta^{(3)}_{1,\delta} &= \frac{6\alpha}{2\alpha+\delta}, \\
\beta^{(3)}_{2,\alpha+\delta} &= \frac{2\alpha(\alpha+3\delta)}{(\alpha+\delta)^2}, \\
\beta^{(3)}_{3,2\alpha+\delta} &= 0.
\end{aligned}$$

The next degree sequence is $d_3 = (0, \delta, \alpha + \delta, \alpha + 2\delta, 2\alpha + 2\delta)$, which has a corresponding elimination matrix $\pi(d_3)$ with entries

$$\begin{aligned}
\pi(d_3)_{1,\delta} &= \frac{1}{\alpha\delta(\alpha+\delta)(2\alpha+\delta)}, \\
\pi(d_3)_{2,\alpha+\delta} &= \frac{1}{\alpha\delta(\alpha+\delta)^2}, \\
\pi(d_3)_{3,\alpha+2\delta} &= \frac{1}{\alpha\delta(\alpha+\delta)(\alpha+2\delta)}.
\end{aligned}$$

So, we need to find the largest x_3 such that $\beta^{(3)} - x_3\pi(d_3)$ has entries that are all ≥ 0. Then we want the largest x_3 such that

$$x_3 \leq 6\alpha^2\delta(\alpha+\delta) = 6\alpha^3\delta + 6\alpha^2\delta^2,$$

$$x_3 \leq 2\alpha^2\delta(\alpha+3\delta) = 2\alpha^3\delta + 6\alpha^2\delta^2,$$

$$x_3 \leq 2\alpha\delta(\alpha+\delta)(\alpha+2\delta) = 2\alpha^3\delta + 6\alpha^2\delta^2 + 4\alpha\delta^3.$$

Observe that $2\alpha^3\delta + 6\alpha^2\delta^2 < 6\alpha^3\delta + 6\alpha^2\delta^2$ and $2\alpha^3\delta + 6\alpha^2\delta^2 < 2\alpha^3\delta + 6\alpha^2\delta^2 + 4\alpha\delta^3$. So, we choose $x_3 = 2\alpha^2\delta(\alpha+3\delta) = 2\alpha^3\delta + 6\alpha^2\delta^2$ as the coefficient for $\pi(d_3)$. Now let $\beta^{(4)} = \beta^{(3)} - 2\alpha^2\delta(\alpha+3\delta)\pi(d_3)$. Then $\beta^{(4)}$ has entries

$$\begin{aligned}
\beta^{(4)}_{1,\delta} &= \frac{6\alpha}{2\alpha+\delta} - \frac{2\alpha^2\delta(\alpha+3\delta)}{\alpha\delta(\alpha+\delta)(2\alpha+\delta)} = \frac{4\alpha^2}{(\alpha+\delta)(2\alpha+\delta)},\\
\beta^{(4)}_{2,\alpha+\delta} &= \frac{2\alpha(\alpha+3\delta)}{(\alpha+\delta)^2} - \frac{2\alpha^2\delta(\alpha+3\delta)}{\alpha\delta(\alpha+\delta)^2} = 0,\\
\beta^{(4)}_{3,\alpha+2\delta} &= 2 - \frac{2\alpha^2\delta(\alpha+3\delta)}{\alpha\delta(\alpha+\delta)(\alpha+2\delta)} = \frac{4\delta^2}{(\alpha+\delta)(\alpha+2\delta)}.
\end{aligned}$$

The final degree sequence is $d_4 = (0, \delta, 2\delta, \alpha+2\delta, 2\alpha+2\delta)$, which has corresponding elimination matrix $\pi(d_4)$ with entries

$$\begin{aligned}
\pi(d_4)_{1,\delta} &= \frac{1}{\delta^2(\alpha+\delta)(2\alpha+\delta)},\\
\pi(d_4)_{2,2\delta} &= \frac{1}{4\alpha^2\delta^2},\\
\pi(d_4)_{3,\alpha+2\delta} &= \frac{1}{\alpha^2(\alpha+\delta)(\alpha+2\delta)}.
\end{aligned}$$

Notice that $4\alpha^2\delta^2\pi(d_4) = \beta^{(4)}$. So we are left with

$$\begin{aligned}
\beta^{(0)} \quad = \quad & 4\alpha^2\delta^2\pi(0, \alpha, 2\alpha, 2\alpha+\delta, 2\alpha+2\delta)\\
& +2\alpha^2\delta(\alpha+3\delta)\pi(0, \alpha, \alpha+\delta, 2\alpha+\delta, 2\alpha+2\delta)\\
& +4\alpha^2\delta(\delta-\alpha)\pi(0, \delta, \alpha+\delta, 2\alpha+\delta, 2\alpha+2\delta)\\
& +2\alpha^2\delta(\alpha+3\delta)\pi(0, \delta, \alpha+\delta, \alpha+2\delta, 2\alpha+2\delta)\\
& +4\alpha^2\delta^2\pi(0, \delta, 2\delta, \alpha+2\delta, 2\alpha+2\delta). \quad \square
\end{aligned}$$

Proposition 3.2.4. *Let $S = k[x, y, z, w]$ be a polynomial ring over a field k and let $I = (x^\alpha, y^\delta, z^\delta, w^\delta)$ be an ideal of S generated by a homogeneous regular sequence. Let $\alpha < \delta$. Then the Betti decomposition of S/I obtained from Algorithm 1.4.5 is given by*

$$\begin{aligned}
\beta(S/I) \quad = \quad & 6\alpha\delta^3\pi(0, \alpha, \alpha+\delta, \alpha+2\delta, \alpha+3\delta) \\
& +6\alpha\delta^3\pi(0, \delta, \alpha+\delta, \alpha+2\delta, \alpha+3\delta) \\
& +6\alpha\delta^3\pi(0, \delta, 2\delta, \alpha+2\delta, \alpha+3\delta) \\
& +6\alpha\delta^3\pi(0, \delta, 2\delta, 3\delta, \alpha+3\delta).
\end{aligned}$$

Proof. Let $S = k[x, y, z, w]$ be a polynomial ring over a field k and let $I = (x^\alpha, y^\delta, z^\delta, w^\delta)$ be an ideal of S generated by a homogeneous regular sequence. Let $\alpha < \delta$. We construct the graded minimal free resolution of S/I as in Remark 1.3.9:

$$S/I \longleftarrow S \longleftarrow \begin{matrix} S(-\alpha) \\ \oplus \\ S^3(-\delta) \end{matrix} \longleftarrow \begin{matrix} S^3(-2\alpha) \\ \oplus \\ S^3(-\alpha-\delta) \end{matrix} \longleftarrow \begin{matrix} S(-\alpha-2\delta) \\ \oplus \\ S(-3\delta) \end{matrix} \longleftarrow S(-\alpha-3\delta) \longleftarrow 0.$$

Then the Betti diagram of S/I, denoted $\beta^{(0)}$, has nonzero entries

$$\beta^{(0)}_{0,0} = 1, \beta^{(0)}_{1,\alpha} = 1, \beta^{(0)}_{1,\delta} = 3, \beta^{(0)}_{2,\alpha+\delta} = 3, \beta^{(0)}_{2,2\delta} = 3, \beta^{(0)}_{3,\alpha+2\delta} = 3, \beta^{(0)}_{3,3\delta} = 1, \beta^{(0)}_{4,\alpha+3\delta} = 1.$$

We follow Algorithm 1.4.5 to find the Betti decomposition of $\beta^{(0)}$. The first degree sequence in the decomposition of $\beta^{(0)}$ is given by $d_0 = (0, \alpha, \alpha+\delta, \alpha+2\delta, \alpha+3\delta)$. Then the nonzero entries of the first elimination matrix are given by $\pi(d_0)$:

$$\begin{aligned}
\pi(d_0)_{1,\alpha} &= \frac{1}{6\alpha\delta^3} \\
\pi(d_0)_{2,\alpha+\delta} &= \frac{1}{2\delta^3(\alpha+\delta)} \\
\pi(d_0)_{3,\alpha+2\delta} &= \frac{1}{2\delta^3(\alpha+2\delta)}.
\end{aligned}$$

We want to construct a new $\beta^{(1)} = \beta^{(0)} - x_0\pi(d_0)$ by substracting a scalar multiple $x_0 \in \mathbb{Z}^+$

of $\pi(d_0)$ from $\beta^{(0)}$ so that the result has entries 0 or greater than 0 and that we eliminate one of the nonzero entries from $\beta^{(0)}$. This scalar will be the largest x_0 that satisfies the following inequalities:

$$\begin{aligned} 1 - \frac{x_0}{6\alpha\delta^3} &\geq 0 \\ 3 - \frac{x_0}{2\delta^3(\alpha+\delta)} &\geq 0 \\ 3 - \frac{x_0}{2\delta^3(\alpha+2\delta)} &\geq 0. \end{aligned}$$

We see that $6\alpha\delta^3$ satisfies the above inequalities. Then $\beta^{(1)} = \beta^{(0)} - 6\alpha\delta^3\pi(d_0)$ has nonzero entries given by

$$\begin{aligned} \beta^{(1)}_{1,\alpha} &= 0 \\ \beta^{(1)}_{1,\delta} &= 3 \\ \beta^{(1)}_{2,\alpha+\delta} &= \frac{3\delta}{\alpha+\delta} \\ \beta^{(1)}_{2,2\delta} &= 3 \\ \beta^{(1)}_{3,\alpha+2\delta} &= \frac{6\delta}{(\alpha+2\delta)} \\ \beta^{(1)}_{3,3\delta} &= 1. \end{aligned}$$

Following Algorithm 1.4.5, we have the degree sequence $d_1 = (0, \delta, \alpha+\delta, \alpha+2\delta, \alpha+3\delta)$ and nonzero entries of the corresponding elimination matrix $\pi(d_1)$:

$$\begin{aligned} \pi(d_1)_{1,\delta} &= \frac{1}{\alpha\delta(\alpha+\delta)(2\alpha+\delta)} \\ \pi(d_1)_{2,\alpha+\delta} &= \frac{1}{2\alpha\delta^2(\alpha+\delta)} \\ \pi(d_1)_{3,\alpha+2\delta} &= \frac{1}{\delta^2(\alpha+\delta)(\alpha+2\delta)}. \end{aligned}$$

We find that the largest x_1 such that each entry of $\beta^{(2)} = \beta^{(1)} - x_1\pi(d_1)$ is greater than or equal to 0 is $6\alpha\delta^3$:

$$\begin{aligned}
3 - \frac{x_1}{\alpha\delta(\alpha+\delta)(\alpha+2\delta)} &\geq 0 \\
\frac{3\delta}{\alpha+\delta} - \frac{x_1}{2\alpha\delta^2(\alpha+\delta)} &\geq 0 \\
\frac{6\delta}{\alpha+2\delta} - \frac{x_1}{\delta^2(\alpha+\delta)(\alpha+2\delta)} &\geq 0.
\end{aligned}$$

Then the nonzero entries of $\beta^{(2)}$ are given by

$$\begin{aligned}
\beta^{(2)}_{1,\delta} &= \frac{3\alpha(\alpha+3\delta)}{(\alpha+\delta)(\alpha+2\delta)} \\
\beta^{(2)}_{2,\alpha+\delta} &= 0 \\
\beta^{(2)}_{2,2\delta} &= 3 \\
\beta^{(2)}_{3,\alpha+2\delta} &= \frac{6\delta^2}{(\alpha+\delta)(\alpha+2\delta)} \\
\beta^{(2)}_{3,3\delta} &= 1
\end{aligned}$$

Then the next degree sequence is $d_2 = (0, \delta, 2\delta, \alpha+2\delta, \alpha+3\delta)$ and the nonzero entries of the elimination matrix $\pi(d_2)$ are

$$\begin{aligned}
\pi(d_2)_{1,\delta} &= \frac{1}{\delta^2(\alpha+\delta)(\alpha+2\delta)} \\
\pi(d_2)_{2,2\delta} &= \frac{1}{2\alpha\delta^2(\alpha+\delta)} \\
\pi(d_2)_{3,\alpha+2\delta} &= \frac{1}{\alpha\delta(\alpha+\delta)(\alpha+2\delta)}.
\end{aligned}$$

We find that $6\delta^3$ is the largest x_2 such that $\beta^{(3)} = \beta^{(2)} - x_2\delta\pi(d_2)$ has entries that are greater than or equal to zero. The nonzero entries of $\beta^{(3)}$ are:

$$\begin{aligned}
\beta^{(3)}_{1,\delta} &= \frac{3\alpha}{(\alpha+2\delta)} \\
\beta^{(3)}_{2,2\delta} &= \frac{6\alpha}{2(\alpha+\delta)} \\
\beta^{(3)}_{3,\alpha+2\delta} &= 0 \\
\beta^{(3)}_{3,3\delta} &= 1
\end{aligned}$$

Then we have our final degree sequence $d_3 = (0, \delta, \alpha+\delta, 2\alpha+\delta, 3\alpha+\delta)$ and the nonzero entries of the corresponding elimination matrix $\pi(d_3)$:

$$\begin{aligned}
\pi(d_3)_{1,\delta} &= \frac{1}{2\delta^3(\alpha+2\delta)} \\
\pi(d_3)_{2,2\delta} &= \frac{1}{2\delta^3(\alpha+\delta)} \\
\pi(d_3)_{3,3\delta} &= \frac{1}{6\alpha\delta^3}.
\end{aligned}$$

Observe that $6\alpha\delta^3\pi(d_3) = \beta^{(3)}$. So we have

$$\begin{aligned}
\beta^{(0)} \quad = \quad & 6\alpha\delta^3\pi(0, \alpha, \alpha+\delta, \alpha+2\delta, \alpha+3\delta) \\
& +6\alpha\delta^3\pi(0, \delta, \alpha+\delta, \alpha+2\delta, \alpha+3\delta) \\
& +6\alpha\delta^3\pi(0, \delta, 2\delta, \alpha+2\delta, \alpha+3\delta) \\
& +6\alpha\delta^3\pi(0, \delta, 2\delta, 3\delta, \alpha+3\delta). \quad \square
\end{aligned}$$

Let $S/I = k[x, y, z, w]/(x^\alpha, y^\alpha, z^\gamma, w^\delta)$ such that $\alpha < \gamma < \delta$. We attempt to construct a general form for the Betti decomposition of S/I using a similar method as with Propositions 3.2.1, 3.2.2, 3.2.3, and 3.2.4, but we find that this strategy alone will not suffice. We begin to find the general Betti decomposition of S/I in order to identify the problem. We construct the resolution of S/I as in Remark 1.3.9:

$$S/I \leftarrow S \leftarrow \begin{array}{c} S^2(-\alpha) \\ \oplus \\ S(-\gamma) \\ \oplus \\ S(-\delta) \end{array} \leftarrow \begin{array}{c} S(-2\alpha) \\ \oplus \\ S^2(-\alpha-\gamma) \\ \oplus \\ S^2(-\alpha-\delta) \\ \oplus \\ S(-\gamma-\delta) \end{array} \leftarrow \begin{array}{c} S(-2\alpha-\gamma) \\ \oplus \\ S(-2\alpha-\delta) \\ \oplus \\ S^2(-\alpha-\gamma-\delta) \end{array} \leftarrow S(-2\alpha-\gamma-\delta) \leftarrow 0,$$

and we have the corresponding nonzero Betti diagram entries,

$$\beta^{(0)}_{0,0} = 1, \beta^{(0)}_{1,\alpha} = 2, \beta^{(0)}_{1,\gamma} = 1, \beta^{(0)}_{1,\delta} = 1,$$

$$\beta^{(0)}_{2,2\alpha} = 1, \beta^{(0)}_{2,\alpha+\gamma} = 2, \beta^{(0)}_{2,\alpha+\delta} = 2, \beta^{(0)}_{2,\gamma+\delta} = 1,$$

$$\beta^{(0)}_{3,2\alpha+\gamma} = 1, \beta^{(0)}_{3,2\alpha+\delta} = 1, \beta^{(0)}_{3,\alpha+\gamma+\delta} = 2, \beta^{(0)}_{4,2\alpha+\gamma+\delta} = 1.$$

Our first degree sequence is $d_0 = (0, \alpha, 2\alpha, 2\alpha+\gamma, 2\alpha+\gamma+\delta)$. The corresponding elimination matrix $\pi(d_0)$ has nonzero entries

$$\begin{aligned}
\pi(d_0)_{1,\alpha} &= \frac{1}{\alpha^2(\alpha+\gamma)(\alpha+\gamma+\delta)} \\
\pi(d_0)_{2,2\alpha} &= \frac{1}{2\alpha^2\gamma(\gamma+\delta)} \\
\pi(d_0)_{3,2\alpha+\gamma} &= \frac{1}{\gamma\delta(\alpha+\gamma)(2\alpha+\gamma)}.
\end{aligned}$$

We need to find the largest x_0 such that $\beta^{(0)} - x_0\pi(d_0)$ has entries that are greater than or equal to 0. So, we need the largest x_0 such that

$$x_0 \leq 2\alpha^2(\alpha+\gamma)(\alpha+\gamma+\delta) = 2\alpha^4 + 4\alpha^3\gamma + 2\alpha^3\delta + 2\alpha^2\gamma^2 + 2\alpha^2\gamma\delta$$

$$x_0 \leq 2\alpha^2\gamma(\gamma+\delta) = 2\alpha^2\gamma^2 + 2\alpha^2\gamma\delta$$

$$x_0 \leq \gamma\delta(\alpha+\gamma)(2\alpha+\gamma) = 2\alpha^2\gamma\delta + 3\alpha\gamma^2\delta + \gamma^3\delta.$$

It is clear that $2\alpha^2(\alpha+\gamma)(\alpha+\gamma+\delta) = 2\alpha^2(\alpha^2 + 2\alpha\gamma + \alpha\delta + \gamma^2 + \gamma\delta) > 2\alpha^2(\gamma^2+\gamma\delta) = 2\alpha^2\gamma(\gamma+\delta)$ since $\alpha^2+2\alpha\gamma+\alpha\delta+\gamma^2+\gamma\delta > \gamma^2+\gamma\delta$. It remains to compare $2\alpha^2\gamma(\gamma+\delta)$ and $\gamma\delta(\alpha+\gamma)(2\alpha+\gamma)$. We do this by subtracting $\gamma\delta(\alpha+\gamma)(2\alpha+\gamma)$ from $2\alpha^2(\alpha+\gamma)(\alpha+\gamma+\delta)$:

$$\begin{aligned}
2\alpha^2\gamma(\gamma+\delta) - \gamma\delta(\alpha+\gamma)(2\alpha+\gamma) &= 2\alpha^2\gamma^2 + 2\alpha^2\gamma\delta - (2\alpha^2\gamma\delta + 3\alpha\gamma^2\delta + \gamma^3\delta) \\
&= 2\alpha^2\gamma^2 + 2\alpha^2\gamma\delta - 2\alpha^2\gamma\delta - 3\alpha\gamma^2\delta - \gamma^3\delta \\
&= 2\alpha^2\gamma^2 - 3\alpha\gamma^2\delta - \gamma^3\delta \\
&= \gamma^2(2\alpha^2 - 3\alpha\delta - \gamma\delta).
\end{aligned}$$

Since $\alpha < \gamma < \delta$, it follows that $2\alpha^2 < 2\alpha\delta$. So $2\alpha^2 < 3\alpha\delta + \gamma\delta$. Therefore $2\alpha^2 - 3\alpha\delta - \gamma\delta < 0$. It follows that $\gamma^2(2\alpha^2 - 3\alpha\delta - \gamma\delta) < 0$, and so $2\alpha^2\gamma(\gamma+\delta) - \gamma\delta(\alpha+\gamma)(2\alpha+\gamma) < 0$. Thus $2\alpha^2\gamma(\gamma+\delta) < \gamma\delta(\alpha+\gamma)(2\alpha+\gamma)$. So, we choose $x_0 = 2\alpha^2\gamma(\gamma+\delta)$ as our coefficient of $\pi(d_0)$. Let $\beta^{(1)} = \beta^{(0)} - 2\alpha^2\gamma(\gamma+\delta)\pi(d_0)$. Then $\beta^{(1)}$ has nonzero entries

$$\begin{aligned}
\beta^{(1)}_{1,\alpha} &= \frac{2\alpha(\alpha+2\gamma+\delta)}{(\alpha+\gamma)(\alpha+\gamma+\delta)} \\
\beta^{(1)}_{2,2\alpha} &= 0 \\
\beta^{(1)}_{3,2\alpha+\gamma} &= \frac{\gamma(3\alpha\delta+\gamma\delta-2\alpha^2)}{\delta(\alpha+\gamma)(2\alpha+\gamma)}.
\end{aligned}$$

Our next degree sequence is $d_1 = (0, \alpha, \alpha+\gamma, 2\alpha+\gamma, 2\alpha+\gamma+\delta)$ and has a corresponding elimination matrix $\pi(d_1)$ with nonzero entries

$$\begin{aligned}
\pi(d_1)_{1,\alpha} &= \frac{1}{\alpha\gamma(\alpha+\gamma)(\alpha+\gamma+\delta)} \\
\pi(d_1)_{2,\alpha+\gamma} &= \frac{1}{\alpha\gamma(\alpha+\gamma)(\alpha+\delta)} \\
\pi(d_1)_{3,2\alpha+\gamma} &= \frac{1}{\alpha\delta(\alpha+\gamma)(2\alpha+\gamma)}.
\end{aligned}$$

Looking for the largest x_1 such that $\beta^{(1)} - x_1\pi(d_1)$ has all entries that are greater than or equal to 0, we need the largest x_1 such that

$$\begin{aligned}
x_0 &\leq 2\alpha^2\gamma(\alpha+2\gamma+\delta) \\
x_0 &\leq 2\alpha\gamma(\alpha+\gamma)(\alpha+\delta) \\
x_0 &\leq \alpha\gamma(3\alpha\delta+\gamma\delta-2\alpha^2).
\end{aligned}$$

Notice that

$$2\alpha^2\gamma(\alpha+2\gamma+\delta) = \alpha\gamma(2\alpha^2+4\alpha\gamma+2\alpha\delta) < \alpha\gamma(2\alpha^2+2\alpha\gamma+2\alpha\delta+2\gamma\delta) = 2\alpha\gamma(\alpha+\gamma)(\alpha+\delta),$$

since $2\alpha\gamma < 2\gamma\delta$. It remains to compare $2\alpha^2\gamma(\alpha+2\gamma+\delta)$ and $\alpha\gamma(3\alpha\delta+\gamma\delta-2\alpha^2)$. We do this by subtracting $\alpha\gamma(3\alpha\delta+\gamma\delta-2\alpha^2)$ from $2\alpha^2\gamma(\alpha+2\gamma+\delta)$:

$$\begin{aligned}
2\alpha^2\gamma(\alpha+2\gamma+\delta)-\alpha\gamma(3\alpha\delta+\gamma\delta-2\alpha^2) &= \alpha\gamma(2\alpha^2+4\alpha\gamma+2\alpha\delta)-\alpha\gamma(3\alpha\delta+\gamma\delta-2\alpha^2)\\
&= \alpha\gamma(4\alpha^2+4\alpha\gamma-\alpha\delta-\gamma\delta)\\
&= 4\alpha^3\gamma+4\alpha^2\gamma^2-\alpha^2\gamma\delta-\alpha\gamma^2\delta\\
&= 4\alpha^2\gamma(\alpha+\gamma)-\alpha\gamma\delta(\alpha+\gamma)\\
&= \alpha\gamma(4\alpha-\delta)(\alpha+\gamma).
\end{aligned}$$

Let's look at the roots of this equation. Since $0 < \alpha < \gamma$, the only way that this expression will be 0 is when $4\alpha-\delta = 0$. So, we must consider three cases: (1) $4\alpha = \delta$, (2) $4\alpha < \delta$, and (3) $4\alpha > \delta$. As it is unclear how to proceed from here, we leave the remaining cases as future work and state the current results.

Corollary 3.2.5. *Let $S = k[x_1,\ldots,x_d]$ be a polynomial ring over a field k and let $I = (f_1,\ldots,f_d)$ be an ideal of S generated by a homogeneous regular sequence with deg$(f_i) = e_i$. If $d \leq 3$, then the Betti decomposition of S/I obtained from Algorithm 1.4.5 is completely determined by the degrees $e_1,\ldots,e_d$. In particular, for*

$d=1: \quad \beta(S/I) = e_1 \cdot \pi(0, e_1)$

$d=2: \quad \beta(S/I) = e_1e_2 \cdot \pi(0, e_1, e_1+e_2) + e_1e_2 \cdot \pi(0, e_2, e_1+e_2)$

$d=3:$ *If* $e_1 \le e_2 \le e_3$, *then*

$$\begin{aligned}\beta(S/I) = \ & e_1e_2(e_2+e_3) \cdot \pi(0, e_1, e_1+e_2, e_1+e_2+e_3) \\ & + e_1e_2(e_3-e_1) \cdot \pi(0, e_2, e_1+e_2, e_1+e_2+e_3) \\ & + 2e_1e_2(e_1+e_3-e_2) \cdot \pi(0, e_2, e_1+e_3, e_1+e_2+e_3) \\ & + e_1e_2(e_3-e_1) \cdot \pi(0, e_3, e_1+e_3, e_1+e_2+e_3) \\ & + e_1e_2(e_2+e_3) \cdot \pi(0, e_3, e_2+e_3, e_1+e_2+e_3)\end{aligned}$$

$d=4:$ *If* $e_1 = e_2 = e_3 = e_4$, *then*

$$\beta(S/I) = 24{e_1}^4\pi(0, e_1, 2e_1, 3e_1, 4e_1)$$

If $e_1 = e_2 = e_3 < e_4$, *then*

$$\begin{aligned}\beta(S/I) = \ & 6{e_1}^3e_4\pi(0, e_1, 2e_1, 3e_1, 3e_1+e_4) \\ & + 6{e_1}^3e_4\pi(0, e_1, 2e_1, 2e_1+e_4, 3e_1+e_4) \\ & + 6{e_1}^3e_4\pi(0, e_1, e_1+e_4, 2e_1+e_4, 3e_1+e_4) \\ & + 6{e_1}^3e_4\pi(0, e_4, e_1+e_4, 2e_1+e_4, 3e_1+e_4)\end{aligned}$$

If $e_1 = e_2 < e_3 = e_4$, *then*

$$\begin{aligned}\beta(S/I) = \ & 4e_1^2e_4^2\pi(0, e_1, 2e_1, 2e_1+e_4, 2e_1+2e_4) \\ & + 2e_1^2e_4(e_1+3e_4)\pi(0, e_1, e_1+e_4, 2e_1+e_4, 2e_1+2e_4) \\ & + 4e_1^2e_4(e_4-e_1)\pi(0, e_4, e_1+e_4, 2e_1+e_4, 2e_1+2e_4) \\ & + 2e_1^2e_4(e_1+3e_4)\pi(0, e_4, e_1+e_4, e_1+2e_4, 2e_1+2e_4) \\ & + 4e_1^2e_4^2\pi(0, e_4, 2e_4, e_1+2e_4, 2e_1+2e_4)\end{aligned}$$

If $e_1 < e_2 = e_3 = e_4$, *then*

$$\begin{aligned}\beta(S/I) = \ & 6e_1e_4^3\pi(0, e_1, e_1+e_4, e_1+2e_4, e_1+3e_4) \\ & + 6e_1e_4^3\pi(0, e_4, e_1+e_4, e_1+2e_4, e_1+3e_4) \\ & + 6e_1e_4^3\pi(0, e_4, 2e_4, e_1+2e_4, e_1+3e_4) \\ & + 6e_1e_4^3\pi(0, e_4, 2e_4, 3e_4, e_1+3e_4).\end{aligned}$$

Proof. This follows from [6], Proposition 3.2.1, Proposition 3.2.2, Proposition 3.2.3, and Proposition 3.2.4. □

4

Further directions

There is no question that this new and exciting area of research still has much to be discovered. In particular, we find that both of our main questions from Section 1.2 remain open. We state these questions here for convenience:

Question 1.2.1. *Let R be a ring. Consider a short exact sequence of R-modules:*

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0.$$

Given the Betti decompositions of A and C, what can we conclude about the Betti decomposition of B?

Question 1.2.2. *Let $S = k[x_1, \ldots, x_d]$ be a polynomial ring over a field k and let $I = (f_1, \ldots, f_d)$ be an ideal of S generated by a homogeneous regular sequence with $deg(f_i) = e_i$. What is the Betti decomposition of S/I in terms of the degrees e_i?*

In Section 2.1, we identify a class of Betti diagrams in which we can find the Betti decomposition of the sum of two Betti diagrams by taking the sum of the Betti decompositions of the two Betti diagrams. This leads us to Proposition 2.2.4, which provides an answer to Question 1.2.1 for modules with Betti diagrams that belong to the specific class of Betti diagrams from Section 2.1. However, it remains to find other classes of Betti diagrams for which the Betti decomposition of the sum of Betti diagrams is the sum of the Betti decompositions of Betti diagrams. These classes would allow us to make analogous propositions to Proposition 2.2.4.

Another interesting direction is to consider the short exact sequence of modules, as presented in Question 1.2.1, when B is not $A \oplus C$. This would require the use of Macaulay2 [8] to examine different modules and their Betti decompositions.

In Section 3.2, we present the Betti decompositions of certain cases of complete intersections in codimension 4. It remains to consider the last four cases of complete intersections in codimension 4:

(i) $e_1 = e_2 < e_3 < e_4$,

(ii) $e_1 < e_2 = e_3 < e_4$,

(iii) $e_1 < e_2 < e_3 = e_4$,

(iv) $e_1 < e_2 < e_3 < e_4$.

As we mentioned in Section 3.2, it is unclear how to proceed with the above cases. Another strategy will need to be employed in order to tackle these cases, as well as cases of complete intersections in higher codimension.

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